Application of Bi-Quaternions in Physics

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This paper introduces a new bi-quaternion notation and applies this notation to electrodynamics. A set of extended MAXWELL equations and other fundamental equations of electrodynamics are derived. By applying the LORENTZ condition, these equations reduce to the classical form.

Additionally the bi-quaternion notation allows a compact formulation of SRT. Furthermore an application of bi-quaternions in other disciplines of physics as mechanics (dynamics) is shown.

Introduction

One of the most emotional disputes in the late nineteenth-century was about the mathematical notation to use with electrodynamics equations[2]. Today's vector notation was not fully developed at that time and many physicists – one of them was James Clerk MAXWELL – promoted the quaternion notation. The quaternion was "invented" in 1843 by Sir William Rowan HAMILTON[6]. Peter Guthrie Tait[11] was the most outstanding promoter of quaternions. On the other side Oliver HEAVISIDE[7] and Josiah Willard GIBBS[13] both decided independently that they could better apply a part of the quaternion number than use the entire number, why they proceeded further with that, what today is called the vector notation. Generally the vector notation used in pre-EINSTEIN[4] electrodynamics uses three-dimensional vectors. The quaternion on the other hand is a four-dimensional number. To make the quaternion compatible to the three-dimensional electrodynamics of MAXWELL, HAMILTON and TAIT, the scalar part of the quaternion was indicated with the prefix 'S' and the vector part with the prefix 'V'. By doing this, the quaternion has been 'vectorized'.
This notation was also used by Maxwell in his Treatise[9], where he published some equations with 'vectorized' quaternion notation. But with applying this prefixes the whole benefit of quaternions is not used. Maxwell didn't use any quaternion calculus. He only converted some summarized end results into 'vectorized' quaternion notation. The quaternion notation was simply not as powerful and suitable as the vector notation.

Some time ago we have shown, that the introduction of bi-quaternions – a complex expansion of quaternions – makes a compact formulation of electrodynamics in spirit of TAIT's and HAMILTON's work possible. This article is a continuation of TAIT's and HAMILTON's ideas and shows the broad applicability of bi-quaternions in physics.

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HAMILTON's Quaternions

A general quaternion has both a scalar (real) and a vector (imaginary) part. In the example below 'a' is the scalar part and 'ib + jc + kd' is the vector part.

$$Q = a + bi + cj + dk \tag{1.1}$$

Where a, b, c, and d are real numbers and i, j, k are so-called Hamilton'ian unit vectors with the magnitude of $\sqrt{-1}$. They satisfy the equations

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.2}$$

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ij = -ji & jk = -kj & ki = -ik \end{array}$$

A nice explanation about the rotation capabilities of the HAMILTON'ian units in a three-dimensional ARGAND diagram was published by GOUGH[5].

A quaternion is a hypercomplex number. The quaternion radii (or magnitude) in four-dimensional space is defined similar as for ordinary complex numbers as:

$$|Q| \equiv \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \tag{1.3}$$

as also shown by WALKER[12]. By introducing a conjugate quaternion number

$$Q^* = a - bi - cj - dk \tag{1.4}$$

the quaternion magnitude is also

$$|Q| \equiv \sqrt{QQ^*} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \; . \tag{1.5}$$

Such a four-dimensional quaternion is very suitable to represent an event in four-dimensional space:

$$X = x_0 + x_1 i + x_2 j + x_3 k \; . \tag{1.6}$$

From now on we use the following **convention for indices and unit vectors**:

- Indices k = 1, 2, 3
- Indices j = 0, 1, 2, 3 = 0, k
- The unit vectors of three-dimensional space are $\vec{e}_1, \vec{e}_2, \vec{e}_3 = \vec{e}_k$
- The HAMILTON'ian units i, j, k are written in *italic* letters
- and the imaginary unit $\mathrm{i} = \sqrt{-1}$ in normal letter.

Then we can summarize the HAMILTON'ian units and the unit vectors to

$$\vec{i} \equiv \left(i\,\vec{e}_1,\; j\,\vec{e}_2,\; k\,\vec{e}_3\right) \tag{1.7}$$

and we find the following notation for a quaternion X_j

$$X_j = x_0 + \vec{i} \cdot \vec{x}_k \tag{1.8}$$

Calculus wit Quaternions

We introduce the following two quaternions

$$X=\left(x_0+\vec{i}\cdot\vec{x}\right) \text{ und } Y=\left(y_0+\vec{i}\cdot\vec{y}\right) \ .$$

A conjugate quaternion X^* is then:

$$X^*=\left(x_0-\vec{i}\cdot\vec{x}\right) \tag{1.9}$$

The scalar multiplication X·Y is:

$$X\cdot Y=Y\cdot X=\left(x_0y_0+\vec{x}\cdot\vec{y}\right) \tag{1.10}$$

The multiplication XY is:

$$\begin{aligned} XY&=\left(x_0y_0-\vec{x}\cdot\vec{y}\right)+\vec{i}\cdot\left(x_0\vec{y}+\vec{x}y_0+\vec{x}\times\vec{y}\right)\\ YX&=\left(x_0y_0-\vec{x}\cdot\vec{y}\right)+\vec{i}\cdot\left(x_0\vec{y}+\vec{x}y_0-\vec{x}\times\vec{y}\right)\end{aligned} \tag{1.11}$$

The magnitude of the quaternion X is:

$$\left|X\right|=\sqrt{X\cdot X}=\sqrt{x_0^2+\vec{x}\cdot\vec{x}} \tag{1.12}$$

Then we have also

$$XX^*=X^*X=x_0^2+\vec{x}\cdot\vec{x}=\left|X\right|=X\cdot X \tag{1.13}$$

The inverse quaternion X^{-1} is

$$\frac{1}{X}=\frac{X^*}{XX^*}=\frac{X^*}{\left|X\right|^2}=\frac{x_0-\vec{i}\cdot i\vec{x}}{x_0^2+\vec{x}\cdot\vec{x}} \tag{1.14}$$

And the division Y/X is:

$$\frac{Y}{X}=\frac{YX^*}{XX^*}=\frac{\left(y_0+\vec{i}\cdot i\vec{y}\right)\left(x_0-\vec{i}\cdot i\vec{x}\right)}{x_0y_0+\vec{x}\cdot\vec{x}} \tag{1.15}$$

Introduction of Bi-Quaternions

An expansion of quaternions to bi-quaternions is made by replacing the real numbers a, b, c and d with complex numbers. A (complex) quaternion – or a bi-quaternion – is then:

$$\mathbb{X} = \left(x_0 + \mathrm{i}\,x^0\right) + \vec{i}\cdot\left(\mathrm{i}\vec{x}_k + \vec{x}^k\right) \tag{1.16}$$

This new introduced quaternion is different from the octionions (known form LIE algebra) since the HAMILTON'ian units *i*, *j*, *k* are still valid and no new imaginary units are introduced. Such a bi-quaternion now represents a superposition of two four-dimensional numbers:

$$\mathbb{X} = \mathbb{X}_j + \mathbb{X}^j = \left(x_0 + \vec{i}\cdot\mathrm{i}\vec{x}_k\right) + \left(\mathrm{i}\,x^0 + \vec{i}\cdot\vec{x}^k\right) \tag{1.17}$$

The later shown examination with this eight-fold number shows, that such a bi-quaternion is very useful for a compact description of electrodynamics and other disciplines in physics.

But first we have to decide whether we use $\mathbb{X}_j$ or $\mathbb{X}^j$ as our representation of a physical four-vector. Easy calculations show, that both lead to the same result. Therefore we are free to choose between $\mathbb{X}_j$ and $\mathbb{X}^j$. To still have a real term, we decide for $\mathbb{X}_j$ and apply this part of the bi-quaternion to physical four-vectors.

As we will se later, all basic equations of classical and relativistic electrodynamics can be written just by applying bi-quaternion multiplications without using any tensor multiplication.

A multiplication of two bi-quaternions produces 16 or 64 terms, which in turn build again a bi-quaternion according to (1.17). The mapping of these terms to physical four-vector $\mathbb{X}_j$ gives in parallel $\mathbb{X}^j$=0.

As we show later, this nullifying of the other four-vectors has it's origin in physical laws of conservation.

The bi-quaternion (1.17) can be classified as:

Incomplete bi-quaternion $\qquad \mathbb{X}_j = x_0 + \vec{i}\cdot\mathrm{i}\vec{x}_k \qquad \mathbb{X}^j = x^0 + \vec{i}\cdot\mathrm{i}\vec{x}^k \qquad (1.18)$

Complete bi-quaternion $\qquad \mathbb{X} = \mathbb{X}_j + \mathbb{X}^j = \left(x_0 + \mathrm{i}\,x^0\right) + \vec{i}\cdot\left(\mathrm{i}\vec{x}_k + \vec{i}\cdot\vec{x}^k\right) \qquad (1.19)$

Calculus with incomplete bi-quaternions

We have the following two incomplete bi-quaternions

$$\mathbb{X} = \left(x_0 + \vec{i}\cdot \mathrm{i}\vec{x}_k\right) \text{ und } \mathbb{Y} = \left(y_0 + \vec{i}\cdot \mathrm{i}\vec{y}_k\right) .$$

A conjugate incomplete bi-quaternion $\mathbb{X}^*$ is then:

$$\mathbb{X}^* = \left(x_0 - \vec{i}\cdot \mathrm{i}\vec{x}_k\right) \tag{1.20}$$

The scalar multiplication of two incomplete bi-quaternions $\mathbb{X}\cdot\mathbb{Y}$ is:

$$\mathbb{X}\cdot\mathbb{Y} = \mathbb{Y}\cdot\mathbb{X} = \left(x_0 y_0 - \vec{x}_k\cdot\vec{y}_k\right) \tag{1.21}$$

and $\mathbb{X}\cdot\mathbb{Y}^*$ is:

$$\mathbb{X}\cdot\mathbb{Y}^* = \mathbb{Y}\cdot\mathbb{X}^* = \left(x_0 y_0 + \vec{x}_k\cdot\vec{y}_k\right) \tag{1.22}$$

The multiplication of two incomplete bi-quaternions $\mathbb{X}\mathbb{Y}$ is:

$$\begin{aligned} \mathbb{X}\mathbb{Y} &= \left(x_0 y_0 + \vec{x}_k\cdot\vec{y}_k\right) + \vec{i}\cdot\left[\mathrm{i}\left(x_0\vec{y}_k + \vec{x}_k y_0\right) - \vec{x}_k\times\vec{y}_k\right] \\ \mathbb{Y}\mathbb{X} &= \left(x_0 y_0 + \vec{x}_k\cdot\vec{y}_k\right) + \vec{i}\cdot\left[\mathrm{i}\left(x_0\vec{y}_k + \vec{x}_k y_0\right) + \vec{x}_k\times\vec{y}_k\right] \end{aligned} \tag{1.23}$$

The magnitude if an incomplete bi-quaternion $\mathbb{X}$ is:

$$\left|\mathbb{X}\right| = \sqrt{\mathbb{X}\cdot\mathbb{X}} = \sqrt{x_0^2 - \vec{x}_k\cdot\vec{x}_k} \tag{1.24}$$

Then we have also

$$\mathbb{X}\mathbb{X}^* = \mathbb{X}^*\mathbb{X} = x_0^2 - \vec{x}_k\cdot\vec{x}_k = \left|\mathbb{X}\right| = \mathbb{X}\cdot\mathbb{X} \tag{1.25}$$

Calculus with complete bi-quaternions

We have the following two complete bi-quaternions

$$\mathbb{X} = \left[x_0 + \mathrm{i}x^0 + \vec{i}\cdot\left(\mathrm{i}x_k + x^k\right)\right] \text{ und } \mathbb{Y} = \left[y_0 + \mathrm{i}y^0 + \vec{i}\cdot\left(\mathrm{i}y_k + y^k\right)\right].$$

where the superscripts and subscripts represent the indices k=1,2,3.

A conjugate complete bi-quaternion $\mathbb{X}^*$ is then:

$$\mathbb{X}^* \equiv \left[\left(x_0 - \mathrm{i}x^0\right) - \vec{i}\cdot\left(\mathrm{i}x_k - x^k\right)\right] \tag{1.26}$$

The scalar multiplication $\mathbb{X}\cdot\mathbb{Y}$ is:

$$\begin{aligned} \mathbb{X}\cdot\mathbb{Y} = \mathbb{Y}\cdot\mathbb{X} = &\left(x_0 y_0 - \vec{x}_k\cdot\vec{y}_k + x^0 y^0 - \vec{x}^k\cdot\vec{y}^k\right) \\ &+\mathrm{i}\left(x_0 y^0 + \vec{x}_k\cdot\vec{y}^k + x^0 y_0 + \vec{x}^k\cdot\vec{y}_k\right) \end{aligned} \tag{1.27}$$

and $\mathbb{X}\cdot\mathbb{Y}^*$ is:

$$\mathbb{X}\cdot\mathbb{Y}^* = \mathbb{Y}\cdot\mathbb{X}^* = x_0 y_0 + \vec{x}_k\cdot\vec{y}_k + x^0 y^0 + \vec{x}^k\cdot\vec{y}^k \tag{1.28}$$

The multiplication $\mathbb{X}\mathbb{Y}$ is:

$$\begin{aligned} \mathbb{X}\mathbb{Y} = &\left(x_0 y_0 + \vec{x}_k\cdot\vec{y}_k - x^0 y^0 - \vec{x}^k\cdot\vec{y}^k\right) + \mathrm{i}\left(x_0 y^0 - x_k y^k + x^0 y_0 - x^k y_k\right) \\ &+\vec{i}\cdot\left[\left(x_0\vec{y}^k - x^0\vec{y}_k + y_0\vec{x}^k - y^0\vec{x}_k\right) + \mathrm{i}\left(x_0\vec{y}_k + x^0\vec{y}^k + y_0\vec{x}_k + y^0\vec{x}^k\right)\right. \\ &\left.+\left(\vec{x}^k\times\vec{y}^k - \vec{x}_k\times\vec{y}_k\right) + \mathrm{i}\left(\vec{x}_k\times\vec{y}^k + \vec{x}^k\times\vec{y}_k\right)\right] \end{aligned} \tag{1.29}$$

The magnitude of a complete bi-quaternion $\mathbb{X}$ is:

$$\left|\mathbb{X}\right|^2 = \sqrt{\mathbb{X}\cdot\mathbb{X}} = \sqrt{\left(x_0 x_0 - \vec{x}_k\cdot\vec{x}_k + x^0 x^0 - \vec{x}^k\cdot\vec{x}^k\right) + 2\mathrm{i}\left(x_0 x^0 + \vec{x}_k\cdot\vec{x}^k\right)} \tag{1.30}$$

The multiplication $\mathbb{X}\mathbb{X}^*$ is:

$$\mathbb{X}\mathbb{X}^* = \left(x_0 x_0 + x^0 x^0 - \vec{x}_k \cdot \vec{x}_k - \vec{x}^k \cdot \vec{x}^k\right) + 2\vec{i} \cdot \left[\left(x_0 \vec{x}^k + x^0 \vec{x}_k\right) + i\left(\vec{x}_k \times \vec{x}^k\right)\right] \quad (1.31)$$

The multiplication $\mathbb{X}\mathbb{Y}^*$ is:

$$\begin{aligned}\mathbb{X}\mathbb{Y}^* = &\left(x_0 y_0 + x^0 y^0 - \vec{x}_k \cdot \vec{y}_k - \vec{x}^k \cdot \vec{y}^k\right) + i\left(x^0 y_0 + x^k y_k - x_0 y^0 - x_k y^k\right) \\ +\vec{i} \cdot &\left[\left(x_0 \vec{y}^k + x^0 \vec{y}_k + y_0 \vec{x}^k + y^0 \vec{x}_k\right) + i\left(x^0 \vec{y}^k + y_0 \vec{x}_k - x_0 \vec{y}_k - y^0 \vec{x}^k\right)\right. \\ &\left.\left(\vec{x}_k \times \vec{y}_k + \vec{x}^k \times \vec{y}^k\right) + i\left(\vec{x}_k \times \vec{y}^k + \vec{x}^k \times \vec{y}_k\right)\right]\end{aligned} \quad (1.32)$$

Derivations

The bi-quaternion NABLA operator is:

$$\nabla_j \equiv \frac{\partial}{\partial \mathbb{X}_j} = \frac{1}{c}\frac{\partial}{\partial t} + i\left(\frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k\right) = \frac{1}{c}\frac{\partial}{\partial t} + \vec{i} \cdot i\vec{\nabla} \quad (1.33)$$

The bi-quaternion d'ALEMBERT operator is:

$$\Delta_j \equiv \left|\nabla_j\right|^2 = \nabla_j^* \left(\nabla_j\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\Delta} \quad (1.34)$$

Total time derivative

The operator for the total time derivative of a bi-quaternion is (see Appendix A):

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla}\right) + \vec{i} \cdot \left[i\left(c\vec{\nabla} + \frac{\vec{\mathbf{v}}}{c} \circ \frac{\partial}{\partial t}\right) - \left(\vec{\mathbf{v}} \times \vec{\nabla}\right)\right] \quad (1.35)$$

The special multiplication symbol $\circ$ indicates, that on applying this operator, the scalar multiplication must be used for the scalar part and the cross product must be used for the vector part.

The operator in equation (1.35) is analogue to the known operator of two-dimensional derivations:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\nabla$$

With the following multiplication we get the same result as with (1.35):

$$\left(\mathbb{V}\nabla\right)\mathbb{A} = \left(\left(\frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla}\right) + \vec{i} \cdot \left[i\left(c\vec{\nabla} + \frac{\vec{\mathbf{v}}}{c} \circ \frac{\partial}{\partial t}\right) - \left(\vec{\mathbf{v}} \times \vec{\nabla}\right)\right]\right)\mathbb{A} \quad (1.36)$$

Applying this to the event vector $\mathbb{X}$ we find

$$\mathbb{V} = \frac{d\mathbb{X}}{dt} = \frac{1}{4}\left(\mathbb{V}\nabla\right)\mathbb{X} \quad (1.37)$$

Bi-Quaternions in four-dimensional space

An **event** $\mathbb{X}$ in four-dimensional space can be expressed directly with a bi-quaternion according to

$$\mathbb{X} \equiv ct + \vec{i} \cdot i\vec{x} \quad . \tag{1.38}$$

According (1.12), the magnitude (distance of an event to fulcrum) is:

$$\left|\mathbb{X}\right| = \sqrt{c^2 t^2 - x^2} \tag{1.39}$$

In special theory of relativity SRT this magnitude is invariant for each inertial system. The same is valid for the differential $d\mathbb{X}$. A division of $d\mathbb{X}$ by c gives another invariant:

$$\frac{1}{c} d\mathbb{X} = \sqrt{dt^2 - \frac{1}{c^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right)} = dt\sqrt{1 - \frac{v^2}{c^2}} = d\tau \tag{1.40}$$

This is the time dilatation known of special relativity. For the differential of an event vector $\mathbb{X}$ we also have

$$d\mathbb{X} = cdt + \vec{i} \cdot id\vec{x} \tag{1.41}$$

and we find the well-known relativistic **four-velocity** $\mathbb{U}$ as:

$$\mathbb{U} = \frac{d\mathbb{X}}{d\tau} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{\vec{i} \cdot i\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \tag{1.42}$$

where $\vec{v}$ is the velocity between two relatively moving inertial systems. The magnitude of the relativistic four-velocity is known to be always equal to speed of light c.

$$\left|\mathbb{U}\right| = c \tag{1.43}$$

From (1.42) we find the well-known factors from special relativity:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} , \qquad \gamma^2 - 1 = \frac{v^2}{c^2}\gamma^2 , \qquad \frac{\gamma^2 - 1}{\gamma^2} = \frac{v^2}{c^2} \quad , \tag{1.44a, b, c}$$

Now we build the total time differential of an event vector $\mathbb{X}$ and we get another, different non-relativistic four-velocity (coordination velocity) $\mathbb{V}$

$$\mathbb{V} = \frac{d\mathbb{X}}{dt} = c + \vec{i} \cdot i\vec{v} \tag{1.45}$$

which is connected to the relativistic four-velocity according to

$$\mathbb{U} = \gamma\mathbb{V} \tag{1.46}$$

The addition of two velocities $\mathbb{V}_1$ and $\mathbb{V}_2$ shall further comply with equation (1.43):

$$\left|\mathbb{V}_1 + \mathbb{V}_2\right| = c \qquad \Rightarrow \qquad \mathbb{V}_1 + \mathbb{V}_2 \neq 2c + \vec{i} \cdot \left(\vec{v}_1 + \vec{v}_2\right) \tag{1.47}$$

In next section we drive the law of velocity addition.

The Lorentz transformation in bi-quaternion form

The relativistic four-velocity $\mathbb{U}$ is especially helpful for the formulation of coordination transformation between relative moving systems S' and S (see Appendix B). Shall $\mathbb{U} = c + \vec{i} \cdot i\vec{u}$ be the relative velocity between S and S', then we have:

$$\mathbb{X}' = \frac{\gamma}{c}\mathbb{U}^*\mathbb{X} \tag{1.48}$$

And inversely we have

$$\mathbb{X} = \frac{\gamma}{c}\mathbb{U}\mathbb{X}' \tag{1.49}$$

And solved:

$$ct' = \gamma\left(ct - \frac{\vec{v}\cdot\vec{x}}{c}\right), \qquad \vec{x}' = \gamma\left(\vec{x} - \vec{v}t\right) \tag{1.50)a, b}$$

$$ct = \gamma\left(ct' + \frac{\vec{v}\cdot\vec{x}}{c}\right), \qquad \vec{x} = \gamma\left(\vec{x}' + \vec{v}t\right) \tag{1.51)a, b}$$

The four-velocities $\mathbb{U}$ and $\mathbb{U}$* are the operators to transform a bi-quaternion (four-vector) from one inertial system into another relative moving inertial system. The general form of the transformation operator of values in inertial system S and S' with the relative velocity $\mathbb{U} = c + \vec{i} \cdot i\vec{u}$ is:

$$\ldots' = \frac{\gamma}{c}\mathbb{U}^* \ldots \tag{1.52}$$

$$\ldots = \frac{\gamma}{c}\mathbb{U}\ldots' \tag{1.53}$$

Beside the normal way via the differentials (1.50) or (1.51), the **theorem of velocity addition** can be derived with above transformation equations.

We have the systems S and S' moving against each other with relative velocity $\mathbb{U}$. A body moves in S' with the velocity $\mathbb{V}' = c + \vec{i} \cdot i\vec{v}'$. Then we have

$$\mathbb{V}'' = \frac{\gamma}{c}\mathbb{U}\mathbb{V}' = \gamma\left(c + \frac{uv'}{c}\right) + \gamma\vec{i} \cdot i\left(\vec{u} + \vec{v}'\right) \tag{1.54}$$

The scalar part in four velocity $\mathbb{V}$ must always be equal to speed of light c. This we achieve in (1.54) with a division by $\mathbb{V}_0''$

$$\mathbb{V} = \frac{c}{\mathbb{V}_0''}\mathbb{V}'' = \frac{1}{\gamma}\frac{1}{1+\frac{uv'}{c^2}}\mathbb{V}'' = c + \vec{i} \cdot i\frac{\vec{u}+\vec{v}'}{1+\frac{uv'}{c^2}} \tag{1.55}$$

and thereof

$$\vec{v} = \frac{\vec{u}+\vec{v}'}{1+\frac{uv'}{c^2}} \tag{1.56}$$

$$\left|\mathbb{U} + \mathbb{V}'\right| = c \tag{1.57}$$

The „normalisation“ with $\mathbb{V}_0''$ has it's origin in the „designation“ of velocity $\mathbb{V}$' outside of S'. It's impossible to measure primed velocity $\mathbb{V}$' in un-primed system S directly.

Bi-quaternion electrodynamics for linear & isotropic medium (vacuum)

The bi-quaternion electric and magnetic field

Analogue to the velocity we define the **bi-quaternion potentials** with the components φ [V] and $\vec{A}$ [Vs / m]:

$$\mathbb{A} \equiv \frac{\varphi}{c} + \vec{i} \cdot i\vec{A} \tag{2.1}$$

or with

$$\vec{A} = \frac{\varphi}{c^2}\vec{v} \tag{2.2}$$

also

$$\mathbb{A} \equiv \frac{\varphi}{c^2}\mathbb{V} = \frac{\varphi}{c^2}\left(c + \vec{i} \cdot i\vec{v}\right) \tag{2.3}$$

Then we have for the derivation of the bi-quaternion potentials (2.1):

$$\nabla\mathbb{A} = \left(\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \vec{\nabla} \cdot \vec{A}\right) + \vec{i} \cdot \left[\frac{i}{c}\left(\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t}\right) - \left(\vec{\nabla} \times \vec{A}\right)\right] \tag{2.4}$$

We use the substitutions

$$s = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t} - \vec{\nabla} \cdot \vec{A}\ , \qquad \vec{E} = \frac{\vec{F}_q}{q} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}\ , \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{2.5)a, b, c}$$

and find the equation for the **electric force field** $\mathbb{E}$ [V / m]:

$$\mathbb{E} = -c\nabla\mathbb{A} = c\,s + \vec{i} \cdot \left(i\vec{E} + c\,\vec{B}\right) \quad \text{or} \quad \mathbb{E} = \vec{i} \cdot \left(i\vec{E} + c\vec{B}\right) \text{ with } s=0 \tag{2.6}$$

and the equation for the **magnetic induction** $\mathbb{B}$ [Vs / m^2]:

$$\mathbb{B} = -i\nabla\mathbb{A} = i\,s + \vec{i} \cdot \left(i\vec{B} - \frac{1}{c}\vec{E}\right) \quad \text{or} \quad \mathbb{B} = \vec{i} \cdot \left(i\vec{B} - \frac{1}{c}\vec{E}\right) \text{ with } s=0 \tag{2.7}$$

and find thereof

$$\mathbb{E} = -ic\mathbb{B}\ , \qquad \mathbb{B} = \frac{i}{c}\mathbb{E} \tag{2.8}$$

Now we look closer to the real scalar term s. It is

$$\nabla \cdot \mathbb{A} = -s = -\frac{1}{c^2}\left[\frac{\partial\varphi}{\partial t} + \vec{\nabla} \cdot \left(\varphi\vec{v}\right)\right] = -\frac{1}{c^2}\frac{d\varphi}{dt} \tag{2.9}$$

s is known as the **LORENTZ condition** (*s* = 0). Thus the LORENTZ condition is a demand for a source-free bi-quaternion potential $\mathbb{A}$, because we have

$$-s = \nabla \cdot \mathbb{A} = 0 \quad \rightarrow \quad \frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \quad \rightarrow \quad \frac{\partial\varphi}{\partial t} + \vec{\nabla} \cdot \left(\varphi\vec{v}\right) = 0 \tag{2.10}$$

It shows, that the LORENTZ condition is a demand for conservation of scalar potentials φ; i.e. the LORENTZ condition is a conservation law. From (2.9) we find further and independently of *s* also the **general LORENTZ condition**:

$$\frac{1}{c^2}\frac{d\varphi}{dt} + \nabla \cdot \mathbb{A} = 0 \tag{2.11}$$

The bi-quaternion potential density

Analogue to the potentials we define the **bi-quaternion charge- and current density** with the components ρ [As / m^3] and $\vec{J}$ [A / m^2]:

$$\mathbb{J} \equiv c\,\rho + \vec{i}\cdot i\vec{J} \tag{2.12}$$

or with

$$\vec{J} = \rho\vec{v} = \gamma\rho_0\vec{v} \tag{2.13}$$

also

$$\mathbb{J} \equiv \rho_0\mathbb{U} = \rho\mathbb{V} = c\rho + \vec{i}\cdot i\rho\vec{v} \tag{2.14}$$

Then we have for the derivation of the current density (2.12)

$$\nabla\mathbb{J} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} + \vec{i}\cdot\left[i\left(c\vec{\nabla}\rho + \frac{1}{c}\frac{\partial\vec{J}}{\partial t}\right) - \left(\vec{\nabla}\times\vec{J}\right)\right] \tag{2.15}$$

Introducing the substitutions

$$\underline{\sigma} = -\frac{\partial\rho}{\partial t} - \vec{\nabla}\cdot\vec{J}\,, \qquad \underline{\vec{A}} = -\mu c\vec{\nabla}\rho - \frac{\mu}{c}\frac{\partial\vec{J}}{\partial t}\,, \qquad \underline{\vec{J}} = \vec{\nabla}\times\vec{J} \tag{2.16)a, b$$

we get the equations for a volume current density $\underline{\mathbb{J}}$ [A / m^3]:

$$\underline{\mathbb{J}} = -i\nabla\mathbb{J} = i\underline{\sigma} + \vec{i}\cdot\left(i\underline{\vec{J}} - \frac{1}{\mu}\underline{\vec{A}}\right) \tag{2.17}$$

and for a potential density $\underline{\mathbb{A}}$ [Vs / m^4]:

$$\underline{\mathbb{A}} = -\mu\nabla\mathbb{J} = \mu\underline{\sigma} + \vec{i}\cdot\left(i\underline{\vec{A}} + \mu\underline{\vec{J}}\right) \tag{2.18}$$

and find thereof

$$\underline{\mathbb{A}} = -i\mu\underline{\mathbb{J}}\,, \qquad \underline{\mathbb{J}} = \frac{i}{\mu}\underline{\mathbb{A}} \tag{2.19}$$

Looking more closely to the real scalar term σ we find

$$\nabla\cdot\mathbb{J} = -\underline{\sigma} = -\frac{\partial\rho}{\partial t} - \vec{\nabla}\cdot\left(\rho\vec{v}\right) = -\frac{1}{c^2}\frac{d\rho}{dt} \tag{2.20}$$

$\underline{\sigma}$ is known from charge conservation ($\underline{\sigma} = 0$). Thus the charge conservation law is the demand for a source-free bi-quaternion potential density $\underline{\mathbb{A}}$, because we have

$$-\underline{\sigma} = \mu\nabla\cdot\mathbb{J} = 0 \quad\rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \quad\rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0 \tag{2.21}$$

This conservation law is analogue to conservation of electric scalar potential φ. With (2.21) we get for the bi-quaternion potential density and volume current density:

$$\underline{\mathbb{A}} = \vec{i}\cdot\left(\underline{\vec{A}} + i\underline{\vec{J}}\right), \qquad \underline{\mathbb{J}} = \vec{i}\cdot\left(i\underline{\vec{J}} - \frac{1}{\mu}\underline{\vec{A}}\right) \tag{2.22}$$

MAXWELL's equations for free charge und free current densities

The following equation directly leads to Maxwell's equations in bi-quaternion form:

$$-\frac{1}{c}\nabla^{*}\mathbb{E}=\nabla^{*}\mathrm{i}\mathbb{B}=\nabla^{*}\nabla\mathbb{A}=\Delta\mathbb{A}=\mu\mathbb{J} \qquad (2.23)$$

or expanded:

$$\mu\mathbb{J}=\frac{1}{c}\left(\vec{\nabla}\cdot\vec{E}-\frac{\partial s}{\partial t}\right)-\mathrm{i}\vec{\nabla}\cdot\vec{B}+\vec{i}\cdot\left[\mathrm{i}\left(\vec{\nabla}\times\vec{B}-\frac{1}{c^{2}}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}s\right)-\frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)\right] \qquad (2.24)$$

respectively

$$\mu\mathbb{J}=\begin{pmatrix}\mu c\rho\\ 0\end{pmatrix}+\mathrm{i}\begin{pmatrix}0\\ \mu\vec{J}\end{pmatrix}=\frac{1}{c}\begin{pmatrix}\vec{\nabla}\cdot\vec{E}-\frac{\partial s}{\partial t}\\ \frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\end{pmatrix}+\mathrm{i}\begin{pmatrix}-\vec{\nabla}\cdot\vec{B}\\ \vec{\nabla}\times\vec{B}-\frac{1}{c^{2}}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}s\end{pmatrix} \qquad (2.25)$$

Similar equations have been published by HONIG[8]. Using (2.25) we can print Maxwell's equations in differential form straight forward:

$$\vec{\nabla}\cdot\vec{B}=0 \qquad (2.26)$$

AMPERE's law
$$\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}=0 \qquad (2.27)$$

Extended COULOMB law
$$\vec{\nabla}\cdot\vec{E}-\frac{\partial s}{\partial t}=\frac{\rho}{\varepsilon} \qquad (2.28)$$

Extended FARADAY law
$$\vec{\nabla}\times\vec{B}-\frac{1}{c^{2}}\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}s=\mu\vec{J} \qquad (2.29)$$

With conservation of electric scalar potential (2.10) , the last two equations reduce to

COULOMB's law
$$\varepsilon\vec{\nabla}\cdot\vec{E}=\rho \qquad (2.30)$$

FARADAY's law
$$\vec{\nabla}\times\vec{B}-\frac{1}{c^{2}}\frac{\partial\vec{E}}{\partial t}=\mu\vec{J} \qquad (2.31)$$

From (2.23) we find also the relation between the electric field and current density

$$\mathbb{J}=-\frac{1}{\mu c}\nabla^{*}\mathbb{E}=-\varepsilon c\nabla^{*}\mathbb{E} \qquad (2.32)$$

The wave equations of potentials and force fields

To derive Maxwell's equations we applied the d'Alembert operator to the electric potentials. This is explicitly:

$$\Delta\,\mathbb{A} = \frac{1}{c}\left(\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \vec{\nabla}^2\varphi\right) + \vec{i}\cdot \mathrm{i}\left(\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A}\right) = \mu\mathbb{J} \tag{2.33}$$

This leads to the wave equations for electric potentials

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \vec{\nabla}^2\varphi = \frac{\rho}{\varepsilon} \tag{2.34}$$

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}^2\vec{A} = \mu\vec{J} \tag{2.35}$$

Applying the derivative to (2.23), we find

$$\nabla\nabla^{*}\mathrm{i}\mathbb{B} = \Delta\mathrm{i}\mathbb{B} = \mu\nabla\mathbb{J} \tag{2.36}$$

and thereof

$$-\left(\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} - \vec{\nabla}^2 s\right) - \vec{i}\cdot\left[\frac{\mathrm{i}}{c}\left(\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \vec{\nabla}^2\vec{E}\right) + \left(\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} - \vec{\nabla}^2\vec{B}\right)\right] =$$

$$\mu\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) + \vec{i}\cdot\mu\left[\mathrm{i}\left(c\vec{\nabla}\rho + \frac{1}{c}\frac{\partial\vec{J}}{\partial t}\right) - \left(\vec{\nabla}\times\vec{J}\right)\right]$$

and finally the wave equations of force fields

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} - \vec{\nabla}^2 s = -\mu\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) = 0 \tag{2.37}$$

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \vec{\nabla}^2\vec{E} = -\mu\left(c^2\vec{\nabla}\rho + \frac{\partial\vec{J}}{\partial t}\right) \tag{2.38}$$

$$\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} - \vec{\nabla}^2\vec{B} = \mu\vec{\nabla}\times\vec{J} \tag{2.39}$$

Bi-quaternion LORENTZ force density and LORENTZ force

We define the bi-quaternion power- and force density with the components $\underline{P}$ $[W / m^3]$ and $\underline{\vec{F}}$ $[N / m^3]$, and get

$$\underline{\mathbb{F}} \equiv \frac{1}{c}\underline{P} + \vec{i}\cdot i\underline{\vec{F}} \tag{2.40}$$

Then we choose the bi-quaternion identities

$$\underline{\mathbb{F}} = \frac{1}{c}\mathbb{J}\mathbb{E} = -i\mathbb{J}\mathbb{B} = -\mathbb{J}\nabla\mathbb{A} = -\rho\mathbb{V}\nabla\mathbb{A} = -\rho\frac{d\mathbb{A}}{dt} \tag{2.41}$$

and use the substitutions (2.10) and (2.13) to get the force density on a charge density ρ, which moves with velocity $\mathbb{V}$ in an external potential field $\mathbb{A}$ as

$$\underline{\mathbb{F}} = \frac{1}{c}\begin{pmatrix}\underline{P}\\ 0\end{pmatrix} + i\begin{pmatrix}0\\ \underline{\vec{F}}\end{pmatrix} = \frac{1}{c}\begin{pmatrix}\rho\vec{v}\cdot\vec{E} + c^2\rho s\\ c^2\rho\vec{B} - \rho\vec{v}\times\vec{E}\end{pmatrix} + i\begin{pmatrix}-\rho\vec{v}\cdot\vec{B}\\ \rho\vec{v}\times\vec{B} + \rho\vec{E} + \rho\vec{v}s\end{pmatrix} \tag{2.42}$$

We have finally

$$\rho\vec{v}\cdot\vec{B} = 0 \tag{2.43}$$

$$\frac{\vec{v}}{c^2}\times\vec{E} = \vec{B} \tag{2.44}$$

extended power density
$$\rho\left(\vec{v}\cdot\vec{E} + c^2 s\right) = \underline{P} \tag{2.45}$$

extended LORENTZ force density
$$\rho\left(\vec{v}\times\vec{B} + \vec{E} + \vec{v}s\right) = \underline{\vec{F}} \tag{2.46}$$

Notable is also $\quad \vec{J}\cdot\vec{B} = 0$ and thereof $\mathbb{J}\cdot\mathbb{E} = \underline{P}$ (2.47)a, b

With conservation of electric scalar potentials (2.10), above equations (2.45) and (2.46) reduce to

Power density (s=0)
$$\rho\vec{v}\cdot\vec{E} = \underline{P} \tag{2.48}$$

LORENTZ force density (s=0)
$$\rho\left(\vec{v}\times\vec{B} + \vec{E}\right) = \underline{\vec{F}} \tag{2.49}$$

For a point charge q in potential field $\mathbb{A}$ we find analogue to (2.40) and (2.41)

$$\mathbb{F} \equiv \frac{i}{c}P + \vec{i}\cdot\vec{F} = -q\mathbb{V}\nabla\mathbb{A} = -q\frac{d\mathbb{A}}{dt} \tag{2.50}$$

extended Power density
$$q\left(\vec{v}\cdot\vec{E} + c^2 s\right) = P \tag{2.51}$$

extended LORENTZ force
$$q\left(\vec{v}\times\vec{B} + \vec{E} + \vec{v}s\right) = \vec{F} \tag{2.52}$$

an wit the conservation of electric scalar potential (2.10)

Power density (s=0)
$$q\vec{v}\cdot\vec{E} = P \tag{2.53}$$

LORENTZ force (s=0)
$$q\left(\vec{v}\times\vec{B} + \vec{E}\right) = \vec{F} \tag{2.54}$$

Bi-quaternion Poynting theorem (energy flow density)

First we define the bi-quaternion energy flow density $\underline{\mathbb{W}}$ with the components of energy density $\underline{w}$ [J / m^3] and energy flow density $\vec{\underline{S}}$ [Js / m^4] as

$$\underline{\mathbb{W}} \equiv \underline{w} + \frac{1}{c}\vec{i} \cdot \vec{\underline{S}} \tag{2.55}$$

Now we consider a current density J and a potential field $\mathbb{A}$ (MAXWELL equations) caused by this current density. We find the equilibrium of both field's force densities by multiplications of bi-quaternion MAXWELL equation (2.23) on both sides with $\nabla\mathbb{A}$. Then, by using (2.41) we get:

$$\frac{1}{\mu}(\Delta\mathbb{A})\nabla\mathbb{A} = \mathbb{J}\nabla\mathbb{A} \quad \Rightarrow \quad \frac{1}{\mu}(\Delta\mathbb{A})\nabla\mathbb{A} + \underline{\mathbb{F}} = 0 \tag{2.56}$$

The calculation (Appendix C) shows for the scalar part:

$$\frac{1}{2}\left(\varepsilon\frac{\partial E^2}{\partial t} + \mu\frac{\partial B^2}{\partial t} + \mu\frac{\partial s^2}{\partial t}\right) + \vec{\nabla} \cdot \frac{1}{\mu}\left(\vec{E} \times \vec{B} - s\,\vec{E}\right) + \vec{E} \cdot \vec{J} + \rho c^2 s = 0 \tag{2.57}$$

Now we insert the substitutions for the energy density $\underline{w}$ and the energy flow density $\vec{\underline{S}}$

$$\underline{w} = \frac{1}{2}\left(\varepsilon\vec{E} \cdot \vec{E} + \frac{1}{\mu}\vec{B} \cdot \vec{B} + \frac{1}{\mu}s^2\right), \qquad \vec{\underline{S}} = \left(\vec{E} \times \frac{1}{\mu}\vec{B} - s\,\vec{E}\right) \tag{2.58)a, b}$$

into (2.57) and use (2.45). Then we get the well known Poynting theorem

$$\frac{\partial \underline{w}}{\partial t} + \vec{\nabla} \cdot \vec{\underline{S}} + \underline{P} = 0 \tag{2.59}$$

Or with (2.59) and (2.55) follows the expanded Poynting theorem (with $s \neq 0$)

$$c\nabla \cdot \underline{\mathbb{W}} + \mathbb{J} \cdot \mathbb{E} = 0 \tag{2.60}$$

Incidentally we may also write for the energy density $\underline{w}$ (1.28)

$$\underline{w} = \frac{1}{2}\varepsilon\mathbb{E} \cdot \mathbb{E}^* \tag{2.61}$$

Inserting (2.32) into (2.41), we find also

$$\underline{\mathbb{F}} = \frac{1}{c}\mathbb{J}\mathbb{E} = -\varepsilon\left(\nabla^*\mathbb{E}\right)\mathbb{E} \tag{2.62}$$

The imaginary vector term is (Appendix C)

$$\frac{1}{c^2}\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + \left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} + \frac{1}{c^2}\left(\vec{\nabla} \times \vec{E}\right) \times \vec{E} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E} \times \vec{B} + s\,\vec{E}\right) + \vec{\nabla} \times s\,\vec{B} \; - \; \vec{\underline{F}} = 0 \tag{2.63}$$

There the first three terms correspond to the divergence of Maxwell stress tensor $\mathbb{T}$:

$$\vec{\nabla} \cdot \mathbb{T} = \frac{1}{c^2}\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + \left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} + \frac{1}{c^2}\left(\vec{\nabla} \times \vec{E}\right) \times \vec{E} \tag{2.64}$$

followed by the partial time derivative of expanded Poynting vector.

Bi-quaternion Lorentz transformation of electromagnetic fields

In (2.14) we have used the relation

$$\mathbb{J} = \rho \mathbb{V} = \gamma \rho_0 \mathbb{V} \tag{2.65}$$

In relativistic case we have explicitly for the transformation of charge and current densities

$$\rho = \gamma \rho_0 \quad \text{und} \quad \rho \vec{v} = \vec{J} = \gamma \vec{J}_0 = \gamma \rho_0 \vec{v} \tag{2.66}$$

The Lorentz transformation of bi-quaternion potentials $\mathbb{A}$ is:

$$\frac{\varphi'}{c} + \vec{i} \cdot i\vec{A}' = \mathbb{A}' = \frac{\gamma}{c} \mathbb{U}^* \mathbb{A} = \gamma \left[\frac{\varphi}{c} - \frac{\vec{u} \cdot \vec{A}}{c} + \vec{i} \cdot \left(i\vec{A} - i\frac{\varphi}{c^2} \vec{u} - \frac{\vec{u} \times \vec{A}}{c} \right) \right] \tag{2.67}$$

and with comparison of coefficients we find:

$$\varphi' = \gamma \left(\varphi - \vec{A} \cdot \vec{u} \right) \tag{2.68}$$

$$\vec{A}' = \gamma \left(\vec{A} - \frac{\varphi}{c^2} \vec{u} \right) \tag{2.69}$$

The Lorentz transformation of bi-quaternion current density $\mathbb{J}$ is:

$$c\rho' + \vec{i} \cdot i\vec{J}' = \mathbb{J}' = \frac{\gamma}{c} \mathbb{U}^* \mathbb{J} = \frac{\gamma}{c} \mathbb{U}^* \left(c\rho + \vec{i} \cdot i\vec{J} \right) = \gamma \left[c\rho - \frac{\vec{u} \cdot \vec{J}}{c} + \vec{i} \cdot \left(i\vec{J} - i\rho\vec{u} - \frac{\vec{u} \times \vec{J}}{c} \right) \right]$$

and again with comparison of coefficients we find:

$$\rho' = \gamma \left(\rho - \frac{\vec{J} \cdot \vec{u}}{c^2} \right) \tag{2.70}$$

$$\vec{J}' = \gamma \left(\vec{J} - \rho \vec{u} \right) \tag{2.71}$$

For the Lorentz transformation of force fields $\vec{E}$ and $\vec{B}$ we have:

$$\mathbb{E}' = \frac{\gamma}{c} \mathbb{U}^* \mathbb{E}$$

$$c\,s' + \vec{i} \cdot \left(i\vec{E}' + c\vec{B}' \right) = \gamma \left[\left(c\,s - \frac{\vec{u}}{c} \cdot \vec{E} + i\vec{u} \cdot \vec{B} \right) + \vec{i} \cdot \left(i\vec{E} + i\vec{u} \times \vec{B} + c\vec{B} - \frac{\vec{u}}{c^2} \times \vec{E} - i\,s\,\vec{u} \right) \right]$$

and thereof

$$s' = \gamma \left(s - \frac{\vec{E} \cdot \vec{u}}{c^2} \right) \tag{2.72}$$

$$\vec{E}' = \gamma \left(\vec{E} + \vec{u} \times \vec{B} - s\,\vec{u} \right) \tag{2.73}$$

$$\vec{B}' = \gamma \left(\vec{B} - \frac{\vec{u}}{c^2} \times \vec{E} \right) \tag{2.74}$$

Dynamics & Kinematics

The bi-quaternion gravity field

In this section we describe the dynamics of moving bodies analogue to previous formulas used for electrodynamics. As a starting point we need two fields, whereof the deeper meaning shall not be discussed now:

$\vec{U}(X)$:	Velocity field (analogue to $\vec{A}$)	[m / s]
$\phi(X)$:	Gravity potential (analogue to φ)	[m^2 / s^2]

These fields form the **kinematic bi-quaternion potential field (velocity field)**

$$\mathbb{U} = \frac{\phi}{c} + \vec{i} \cdot \mathrm{i}\vec{U} \tag{3.1}$$

Then we have for the derivation of $\mathbb{U}$

$$\nabla\mathbb{U} = \left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{U}\right) + \vec{i}\cdot\left[\frac{\mathrm{i}}{c}\left(\vec{\nabla}\phi + \frac{\partial\vec{U}}{\partial t}\right) - \left(\vec{\nabla}\times\vec{U}\right)\right] \tag{3.2}$$

Defining the following substitutions:

$$s_m = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} - \vec{\nabla}\cdot\vec{U}\,, \qquad \vec{G} = \frac{\vec{F}_m}{m} = -\vec{\nabla}\phi - \frac{\partial\vec{U}}{\partial t}\,, \qquad \vec{T} = \vec{\nabla}\times\vec{U} \tag{3.3)a, b, c}$$

with

$\vec{F}_m$:	Force on a point mass m (test mass)	[N] = [kg m / s^2]
$\vec{G}$:	Gravity field, acceleration field	[m / s^2]
$\vec{T}$:	Rotations field, dynamic induction	[1 / s]

whereas the gravity field consists of two (established) parts:

$$\vec{G}_G = -\vec{\nabla}\phi \qquad \text{and} \qquad \vec{G}_T = -\frac{\partial\vec{U}}{\partial t} \tag{3.4)a, b}$$

$\vec{G}_G$:	Gravity field of a gravity potential	[m / s^2]
$\vec{G}_T$:	Acceleration field, induced gravity field	[m / s^2]

Then we find for the bi-quaternion gravity field $\mathbb{G}$ [m / s^2] on a test mass m

$$\mathbb{G} = -c\nabla\mathbb{U} = c\,s_m + \vec{i}\cdot\left(\mathrm{i}\vec{G} + c\,\vec{T}\right) \tag{3.5}$$

and for the dynamic induction $\mathbb{T}$ [s^{-1}] we have:

$$\mathbb{T} = -\mathrm{i}\nabla\mathbb{U} = \mathrm{i}\,s_m + \vec{i}\cdot\left(\mathrm{i}\vec{T} - \frac{1}{c}\vec{G}\right) \tag{3.6}$$

Thereof we find

$$\mathbb{G} = -\mathrm{i}c\mathbb{T}\,, \qquad \mathbb{T} = \frac{\mathrm{i}}{c}\mathbb{G} \tag{3.7}$$

The scalar term of (3.2) is – if nullified – again a conservation law:

$$-s_m = \nabla\cdot\mathbb{U} = 0 \quad\rightarrow\quad \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{U} = 0 \quad\rightarrow\quad \frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\left(\phi\vec{U}\right) = 0 \tag{3.8}$$

Bi-quaternion momentum density

Analogue to charge- and current density of electrodynamics we define the bi-quaternion mass- and momentum density $\underline{\mathbb{p}}_m$ with the components $\underline{m}$ [kg / m^3] and $\underline{\vec{p}}_m$ [kg / sm^2]:

$$\underline{\mathbb{p}}_m \equiv c\,\underline{m} + \vec{i} \cdot i\underline{\vec{p}}_m \tag{3.9}$$

or with

$$\underline{\vec{p}}_m = \underline{m}\vec{v} = \gamma\underline{m}_0\vec{v} \tag{3.10}$$

also

$$\underline{\mathbb{p}}_m \equiv \underline{m}\mathbb{V} = \underline{m}c + \vec{i} \cdot i\underline{m}\vec{v} \tag{3.11}$$

Then we have for the derivation of the momentum density [kg / sm^3]:

$$\nabla\underline{\mathbb{p}}_m = \frac{\partial\underline{m}}{\partial t} + \vec{\nabla} \cdot \underline{\vec{p}}_m + \vec{i} \cdot \left[i\left(c\vec{\nabla}\underline{m} + \frac{1}{c}\frac{\partial\underline{\vec{p}}_m}{\partial t} \right) - \left(\vec{\nabla} \times \underline{\vec{p}}_m \right) \right] \tag{3.12}$$

Within scalar part we identify the continuity equation of mass density and the conservation law of mass respectively.

$$\frac{\partial\underline{m}}{\partial t} + \vec{\nabla} \cdot \underline{\vec{p}}_m = 0$$

Instead of mass density c$\underline{m}$ we could also use the energy density $\underline{w}_m$/c. Equation (3.9) changes then to:

$$\underline{\mathbb{p}}_m = \frac{\underline{w}_m}{c} + \vec{i} \cdot i\underline{\vec{p}}_m \tag{3.13}$$

Using point masses instead of mass densities we also have

$$\mathbb{P}_m = \frac{W_m}{c} + \vec{i} \cdot i\vec{p} \tag{3.14}$$

The relativistic momentum $\mathbb{p}_m$ of a point mass m with bi-quaternion velocity $\mathbb{V}$ is

$$\mathbb{P}_m = m_0\gamma\mathbb{V} \tag{3.15}$$

This contemplation is analogue to (3.59) and corresponds with the total self-energy or self-momentum of a mass (or charge, respectively). From (3.15) we find immediately

$$W_m = \gamma m_0 c^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \ldots \tag{3.16}$$

Field equations of dynamics

Equipped with the bi-quaternion momentum density $\mathbb{p}_m$ and the potential field $\mathbb{U}$ we could start again analogue to electrodynamics (2.23):

$$-\frac{1}{c}\nabla^*\mathbb{G} = \nabla^* i\mathbb{T} = \nabla^*\nabla\mathbb{U} = \Delta\mathbb{U} = \frac{g}{c^2}\underline{\mathbb{p}}_m \tag{3.17}$$

where g is the gravity constant with $6.67\cdot10^{-11}$ [N m^2 / kg^2]. From (3.17) we have

$$\frac{g}{c^2}\underline{\mathbb{p}}_m = \frac{1}{c}\left(\vec{\nabla}\cdot\vec{G} - \frac{\partial s_m}{\partial t}\right) - i\vec{\nabla}\cdot\vec{T} + \vec{i}\cdot\left[i\left(\vec{\nabla}\times\vec{T} - \frac{1}{c^2}\frac{\partial\vec{G}}{\partial t} + \vec{\nabla}s_m\right) - \frac{1}{c}\left(\frac{\partial\vec{T}}{\partial t} + \vec{\nabla}\times\vec{G}\right)\right] \tag{3.18}$$

respectively

$$\frac{g}{c^2}\underline{\mathbb{p}}_m = \frac{g}{c^2}\left[\begin{pmatrix} c\underline{m} \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ \underline{\vec{p}} \end{pmatrix}\right] = \frac{1}{c}\begin{pmatrix} \vec{\nabla}\cdot\vec{G} - \dfrac{\partial s_m}{\partial t} \\ \dfrac{\partial\vec{T}}{\partial t} + \vec{\nabla}\times\vec{G} \end{pmatrix} + i\begin{pmatrix} -\vec{\nabla}\cdot\vec{T} \\ \vec{\nabla}\times\vec{T} - \dfrac{1}{c^2}\dfrac{\partial\vec{G}}{\partial t} + \vec{\nabla}s_m \end{pmatrix} \tag{3.19}$$

Thereof we extract the „Maxwell“ equations of dynamics:

$$\vec{\nabla}\cdot\vec{T} = 0 \tag{3.20}$$

„AMPERE‘s law“

$$\frac{\partial\vec{T}}{\partial t} + \vec{\nabla}\times\vec{G} = 0 \tag{3.21}$$

Expanded „COULOMB law“

$$\vec{\nabla}\cdot\vec{G} - \frac{\partial s_m}{\partial t} = g\underline{m} \tag{3.22}$$

Expanded „FARADAY law“

$$\vec{\nabla}\times\vec{T} - \frac{1}{c^2}\frac{\partial\vec{G}}{\partial t} + \vec{\nabla}s_m = \frac{g}{c^2}\underline{\vec{p}} \tag{3.23}$$

With the conservation of gravity potential (3.8), the last two equations reduce to

„COULOMB‘s law“

$$\frac{1}{g}\vec{\nabla}\cdot\vec{G} = \underline{m} \tag{3.24}$$

„FARADAY‘s law“

$$\vec{\nabla}\times\vec{T} - \frac{1}{c^2}\frac{\partial\vec{G}}{\partial t} = \frac{g}{c^2}\underline{\vec{p}} \tag{3.25}$$

From (3.17) we also find the relation between the gravity field $\mathbb{G}$ and the momentum density $\underline{\mathbb{p}}$m

$$\underline{\mathbb{p}}_m = -\frac{c}{g}\nabla^*\mathbb{G} \tag{3.26}$$

The wave equations of the potentials

To derive above equations we implicitly applied the d'Alembert operator to the potentials. Explicitly this is:

$$\Delta \mathbb{U} = \frac{1}{c}\left(\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi\right) + \vec{i}\cdot \mathrm{i}\left(\frac{1}{c^2}\frac{\partial^2 \vec{U}}{\partial t^2} - \vec{\nabla}^2 \vec{U}\right) = \frac{g}{c^2}\underline{\mathbb{P}}_m \tag{3.27}$$

Thereof follows the wave equations of the potentials

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \vec{\nabla}^2 \phi = g\underline{m} \tag{3.28}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{U}}{\partial t^2} - \vec{\nabla}^2 \vec{U} = \frac{g}{c^2}\underline{\vec{p}} \tag{3.29}$$

With the derivation of (3.17) we find

$$\nabla\nabla^* \mathrm{i}\mathbb{T} = \Delta \mathrm{i}\mathbb{T} = \frac{g}{c^2}\nabla\underline{\mathbb{P}}_m \tag{3.30}$$

and thereof

$$-\left(\frac{1}{c^2}\frac{\partial^2 s_m}{\partial t^2} - \vec{\nabla}^2 s_m\right) - \vec{i}\cdot\left[\frac{\mathrm{i}}{c}\left(\frac{1}{c^2}\frac{\partial^2 \vec{G}}{\partial t^2} - \vec{\nabla}^2 \vec{G}\right) + \left(\frac{1}{c^2}\frac{\partial^2 \vec{T}}{\partial t^2} - \vec{\nabla}^2 \vec{T}\right)\right] =$$

$$\frac{g}{c^2}\left(\frac{\partial \underline{m}}{\partial t} + \vec{\nabla}\cdot\underline{\vec{p}}\right) + \vec{i}\cdot\frac{g}{c^2}\left[\mathrm{i}\left(c\vec{\nabla}\underline{m} + \frac{1}{c}\frac{\partial \underline{\vec{p}}}{\partial t}\right) - \left(\vec{\nabla}\times\underline{\vec{p}}\right)\right]$$

And finally we have the wave equations for the force fields

$$\frac{1}{c^2}\frac{\partial^2 s_m}{\partial t^2} - \vec{\nabla}^2 s_m = -\frac{g}{c^2}\left(\frac{\partial \underline{m}}{\partial t} + \vec{\nabla}\cdot\underline{\vec{p}}\right) = 0 \tag{3.31}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{G}}{\partial t^2} - \vec{\nabla}^2 \vec{G} = -g\left(\vec{\nabla}\underline{m} + \frac{1}{c^2}\frac{\partial \underline{\vec{p}}}{\partial t}\right) \tag{3.32}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{T}}{\partial t^2} - \vec{\nabla}^2 \vec{T} = \frac{g}{c^2}\vec{\nabla}\times\underline{\vec{p}} \tag{3.33}$$

Analogue to electrodynamics without charge- and current densities, these equations describe a transverse, gravity wave in the absence of mass densities, having the propagation velocity equal to the speed of light c.

$$\frac{1}{c^2}\frac{\partial^2 \vec{G}}{\partial t^2} - \vec{\nabla}^2 \vec{G} = 0 \tag{3.34}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{T}}{\partial t^2} - \vec{\nabla}^2 \vec{T} = 0 \tag{3.35}$$

The bi-quaternion force density and inertia (reaction force)

With (2.40) we use again the bi-quaternion power- and force density $\underline{\mathbb{F}}$ with the components $\underline{\mathrm{P}}$ [W / m^3] and $\underline{\vec{\mathrm{F}}}$ [N / m^3]

$$\underline{\mathbb{F}} \equiv \frac{1}{c}\underline{\mathrm{P}} + \vec{i}\cdot \mathrm{i}\underline{\vec{\mathrm{F}}}$$

and choose the bi-quaternion equation

$$\underline{\mathbb{F}} = \frac{1}{c}\underline{\mathbb{p}}_{\mathrm{m}}\mathbb{G} = -\mathrm{i}\underline{\mathbb{p}}_{\mathrm{m}}\mathbb{T} = -\underline{\mathbb{p}}_{\mathrm{m}}\nabla\mathbb{U} = -\underline{\mathrm{m}}\mathbb{V}\nabla\mathbb{U} = -\underline{\mathrm{m}}\frac{\mathrm{d}\mathbb{U}}{\mathrm{dt}} \tag{3.36}$$

Using the substitutions (3.8) and (3.10), we find the force density on a mass density $\underline{\mathrm{m}}$, which is moving with velocity $\mathbb{V}$ in an external potential field $\mathbb{U}$ according to

$$\underline{\mathbb{F}} = \frac{1}{c}\begin{pmatrix}\underline{\mathrm{P}}\\ 0\end{pmatrix} + \mathrm{i}\begin{pmatrix}0\\ \underline{\vec{\mathrm{F}}}\end{pmatrix} = \frac{1}{c}\begin{pmatrix}\underline{\mathrm{m}}\vec{\mathrm{v}}\cdot\vec{\mathrm{G}} + \underline{\mathrm{m}}c^2\, s_{\mathrm{m}}\\ \underline{\mathrm{m}}c^2\vec{\mathrm{T}} - \underline{\mathrm{m}}\vec{\mathrm{v}}\times\vec{\mathrm{G}}\end{pmatrix} + \mathrm{i}\begin{pmatrix}-\underline{\mathrm{m}}\vec{\mathrm{v}}\cdot\vec{\mathrm{T}}\\ \underline{\mathrm{m}}\vec{\mathrm{v}}\times\vec{\mathrm{T}} + \underline{\mathrm{m}}\vec{\mathrm{G}} + \underline{\mathrm{m}}\vec{\mathrm{v}}\, s_{\mathrm{m}}\end{pmatrix} \tag{3.37}$$

and finally

$$\underline{\mathrm{m}}\vec{\mathrm{v}}\cdot\vec{\mathrm{T}} = 0 \tag{3.38}$$

$$\frac{\vec{\mathrm{v}}}{c^2}\times\vec{\mathrm{G}} = \vec{\mathrm{T}} \tag{3.39}$$

Expanded power density $$\underline{\mathrm{m}}\left(\vec{\mathrm{v}}\cdot\vec{\mathrm{G}} + c^2\, s_{\mathrm{m}}\right) = \underline{\mathrm{P}} \tag{3.40}$$

Expanded reaction force density $$\underline{\mathrm{m}}\left(\vec{\mathrm{v}}\times\vec{\mathrm{T}} + \vec{\mathrm{G}} + \vec{\mathrm{v}}\, s_{\mathrm{m}}\right) = \underline{\vec{\mathrm{F}}} \tag{3.41}$$

Notable is also $\underline{\vec{\mathrm{p}}}\cdot\vec{\mathrm{T}} = 0$ and thus $\underline{\mathbb{p}}_{\mathrm{m}}\cdot\mathbb{G} = \underline{\mathrm{P}}$ (3.42)a, b

With conservation of gravity potential (3.8), equations (3.40) and (3.41) reduce to

Power density (s_{m}=0) $$\underline{\mathrm{m}}\vec{\mathrm{v}}\cdot\vec{\mathrm{G}} = \underline{\mathrm{P}} \tag{3.43}$$

Reaction force density (s_{m}=0) $$\underline{\mathrm{m}}\left(\vec{\mathrm{v}}\times\vec{\mathrm{T}} + \vec{\mathrm{G}}\right) = \underline{\vec{\mathrm{F}}} \tag{3.44}$$

Analogue to (2.40) and (3.36) we find for a point mass m in a potential field $\mathbb{U}$

$$\mathbb{F} \equiv \frac{\mathrm{i}}{c}\mathrm{P} + \vec{i}\cdot\vec{\mathrm{F}} = -\mathrm{m}\mathbb{V}\nabla\mathbb{U} = -\mathrm{m}\frac{\mathrm{d}\mathbb{U}}{\mathrm{dt}} \tag{3.45}$$

Expanded power $$\mathrm{m}\left(\vec{\mathrm{v}}\cdot\vec{\mathrm{G}} + c^2\, s_{\mathrm{m}}\right) = \mathrm{P} \tag{3.46}$$

Expanded reaction force $$\mathrm{m}\left(\vec{\mathrm{v}}\times\vec{\mathrm{T}} + \vec{\mathrm{G}} + \vec{\mathrm{v}}\, s_{\mathrm{m}}\right) = \vec{\mathrm{F}} \tag{3.47}$$

and with the conservation of gravity potential (3.8) this reduces to

Power (s_{m}=0) $$\mathrm{m}\vec{\mathrm{v}}\cdot\vec{\mathrm{G}} = \mathrm{P} \tag{3.48}$$

Reaktions-Kraft (s_{m}=0) $$\mathrm{m}\left(\vec{\mathrm{v}}\times\vec{\mathrm{T}} + \vec{\mathrm{G}}\right) = \vec{\mathrm{F}} \tag{3.49}$$

Bi-quaternion acceleration and inertia (action force)

The action force to accelerate a point mass m is opposite to inertia force (reaction force) (3.45). The external velocity field $\mathbb{U}$ is replaced by the coordinate velocity $\mathbb{V}$ and we have the equation

$$\mathbb{F}_m = m\mathbb{a} = m\frac{d\mathbb{V}}{dt} = m\mathbb{V}\nabla\mathbb{V} \tag{3.50}$$

The calculation of $\mathbb{a}$ yields:

$$\mathbb{a} = c\begin{pmatrix} e \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ \vec{a} \end{pmatrix} = c\begin{pmatrix} \vec{\nabla}\cdot\vec{v} + \dfrac{\vec{v}}{c^2}\cdot\dfrac{\partial\vec{v}}{\partial t} \\ \vec{\nabla}\times\vec{v} - \dfrac{\vec{v}}{c^2}\times\dfrac{\partial\vec{v}}{\partial t} \end{pmatrix} + i\begin{pmatrix} -\vec{v}\cdot\left(\vec{\nabla}\times\vec{v}\right) \\ \dfrac{\partial\vec{v}}{\partial t} + \vec{v}\left(\vec{\nabla}\cdot\vec{v}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) \end{pmatrix} \tag{3.51}$$

and thereof

$$\vec{v}\cdot\left(\vec{\nabla}\times\vec{v}\right) = 0 \tag{3.52}$$

$$\frac{\vec{v}}{c^2}\times\frac{\partial\vec{v}}{\partial t} = \vec{\nabla}\times\vec{v} \tag{3.53}$$

Flow rate e [s^{-1}]:

$$\vec{\nabla}\cdot\vec{v} + \frac{\vec{v}}{c^2}\cdot\frac{\partial\vec{v}}{\partial t} = e \tag{3.54}$$

Acceleration [m / s^2]:

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\left(\vec{\nabla}\cdot\vec{v}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = \vec{a} \tag{3.55a}$$

or

$$\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}\left(\frac{v^2}{2}\right) - \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) = \vec{a} \tag{3.55b}$$

or

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} - \left(\vec{v}\times\vec{\nabla}\right)\times\vec{v} = \vec{a} \tag{3.55c}$$

Equation (3.55)b is known from fluid mechanics. Additionally we have newly found an equation for the flow rate e with the unit s^{-1}.

To find the force equations, we multiply (3.49) with a point mass m or a mass density <u>m</u>. For example we find for a point mass m

Action force:

$$\vec{F} = m\vec{a} \tag{3.56}$$

Power:

$$P = mc^2 e = mc^2\left(\vec{\nabla}\cdot\vec{v} + \frac{\vec{v}}{c^2}\cdot\frac{\partial\vec{v}}{\partial t}\right) \tag{3.57}$$

Bi-quaternion self-energy density and mass density

Calculating the scalar product between bi-quaternion current density $\mathbb{J}$ and of associated bi-quaternion potential field $\mathbb{A}$, we find the self-energy density $\underline{\mathbb{W}}$ as:

$$\underline{\mathbb{W}} = \mathbb{A} \cdot \mathbb{J} = \frac{\varphi}{c^2} \rho \mathbb{V} \cdot \mathbb{V} \tag{3.58}$$

With the bi-quaternion momentum density $\underline{\mathbb{p}}$ of a mass density $\underline{m}$:

$$\underline{\mathbb{p}} = \underline{m}\mathbb{V} \tag{3.59}$$

we find

$$\underline{\mathbb{W}} = \mathbb{A} \cdot \mathbb{J} = \underline{\mathbb{p}} \cdot \mathbb{V} \tag{3.60}$$

Thereof and with (2.34) follows for a mass density and for static case ($\partial\varphi/\partial t=0$) only:

$$\underline{m} = \frac{\varphi}{c^2} \rho = -\frac{\varepsilon}{c^2} \varphi\Delta\varphi \tag{3.61}$$

The integration over the whole volume V is

$$m = \int \underline{m}\ dV = -\frac{\varepsilon}{c^2} \int \varphi\Delta\varphi\ dV = \frac{\varepsilon}{c^2} \int \left|\nabla\varphi\right|^2\ dV = \frac{\varepsilon}{c^2} \int \left(\vec{E} \cdot \vec{E}\right) dV \tag{3.62}$$

For a spherical potential field this becomes

$$m = \int_r^\infty \frac{q^2}{4\pi\varepsilon c^2} \frac{1}{r_m^2} dr = \frac{q^2}{4\pi\varepsilon c^2} \frac{1}{r_m} \tag{3.63}$$

With the elementary charge $q=1.602 \cdot 10^{-19}$ [As] and the electron mass $m_e=9.11 \cdot 10^{-31}$ [kg] we find for example the following value for the electron radius r_e:

$$r_e = \frac{q^2}{4\pi\varepsilon c^2} \frac{1}{m_e} = 2.818 \cdot 10^{-15} \left[m\right] \tag{3.64}$$

From (3.62) and with $\partial\varphi/\partial t=0$ we also find the relation

$$c^2 \int \underline{m}\ dV = \varepsilon \int \left(\vec{E} \cdot \vec{E}\right) dV \tag{3.65}$$

Summary

All important basic equations of electrodynamics can be cast in a very compact form by using bi-quaternions. Beside the known text-book equations we find some possible expansions to Maxwell's equations and other fundamental equations of electrodynamics. This expanded equations reduce to classical form if the LORENTZ condition s=0 is applied.

An interpretation of the scalar field s beyond the interpretation as pure LORENTZ condition shall be discussed at another place.

Beside electrodynamics, bi-quaternions are also very useful for application in other disciplines of physics as for example in mechanics (dynamics) or – as only indicated in the appendix – in quantum mechanics.

References

[1] VAN VLAENDEREN Koen and André WASER, "Generalisation of classical electrodynamics to admit a scalar field and longitudinal waves", *Hadronic Journal* 24 (2001) 609-628

[2] BORK Alfred M., "Vectors Versus Quaternions – The Letters of Nature", *American Journal of Physics* **34** (1966) 202-211

[3] CONWAY A. W., "Quaternion Treatment of the Relativistic Wave Equation", *Proceedings of the Royal Society London, Series A* **162** (15 September 1937) 145-154

[4] EINSTEIN Albert, „Zur Elektrodynamik bewegter Körper", *Annalen der Physik und Chemie* **17** (30. Juni 1905) 891-921

[5] GOUGH W., „Quaternions and spherical harmonics", *European Journal of Physics* **5** (1984) 163-171

[6] HAMILTON William Rowan, "On a new Species of Imaginary Quantities connected with a theory of Quaternions", *Proceedings of the Royal Irish Academy* **2** (13 November 1843) 424-434

[7] HEAVISIDE Oliver, „On the Forces, Stresses and Fluxes of Energy in the Electromagnetic Field", *Philosophical Transactions of the Royal Society* **183A** (1892) 423

[8] HONIG William M., "Quaternionic Electromagnetic Wave Equation and a Dual Charge-Filled Space", *Lettere al Nuovo Cimento, Ser. 2* **19** No.4 (28 Maggio 1977) 137-140

[9] MAXWELL James Clerk, „A Treatise on Electricity & Magnetism", (1893) *Dover Publications, New York* ISBN 0-486-60636-8 (Vol. 1) & 0-486-60637-6 (Vol. 2)

[10] SILBERSTEIN L., "Quaternionic Form of Relativity", *Philosophical Magazine, Ser. 6* **23** (1912) 790-809

[11] TAIT Peter Guthrie, "An elementary Treatise on Quaternions", *Oxford University Press* 1st Edition (1875)

[12] WALKER Marshall J., "Quaternions as 4-Vectors", *American Journal of Physics* 24 (1956) 515-522

[13] WILSON E. B., "Vector Analysis of Josiah Willard Gibbs – The History of a Great Mind", *Charles Scribner's Sons* New York (1901)

Appendix A

We have a bi-quaternion field $\mathbb{A} = \mathbb{A}(\mathbb{X})$, where $\mathbb{X}$ is a function of space and time, and we have $\mathbb{A} = \mathbb{A}(\mathbb{X}(t, x_1, x_2, x_3))$. The total time derivation of $\mathbb{A}$ is then:

$$d\mathbb{A} = \frac{\partial \mathbb{A}(\mathbb{X})}{\partial \mathbb{X}} d\mathbb{X} = \frac{\partial \mathbb{A}}{\partial \mathbb{X}} \frac{\partial \mathbb{X}}{\partial t} dt + \frac{\partial \mathbb{A}}{\partial \mathbb{X}} \frac{\partial \mathbb{X}}{\partial \vec{\mathbf{x}}} d\vec{\mathbf{x}} = -\left(\frac{\partial \mathbb{X}}{\partial t} \frac{\partial \mathbb{A}}{\partial \mathbb{X}} dt + \frac{\partial \mathbb{X}}{\partial \vec{\mathbf{x}}} \frac{\partial \mathbb{A}}{\partial \mathbb{X}} d\vec{\mathbf{x}} \right) \quad \text{(A.1)}$$

with

$$\frac{\partial}{\partial \mathrm{X}} = \nabla \quad \text{and} \quad \frac{\partial}{\partial \vec{\mathbf{x}}} = \vec{\nabla} \quad \text{(A.2)}$$

Now we expand the operator in first term of (A.1) and get

$$\frac{\partial \mathbb{X}}{\partial t} \frac{\partial}{\partial \mathbb{X}} dt = \left(\frac{\partial}{\partial t} + ic\,\vec{i} \cdot \vec{\nabla} \right) dt \quad \text{(A.3)}$$

The operator in second term of (A.1) has the following expansion:

$$\begin{aligned}
\frac{\partial \mathbb{X}}{\partial \vec{\mathbf{x}}} \frac{\partial}{\partial \mathbb{X}} d\vec{\mathbf{x}} \quad &= \left\{ \frac{\partial \mathbb{X}}{\partial \vec{\mathbf{x}}} d\vec{\mathbf{x}} \right\} \frac{\partial}{\partial \mathbb{X}} \\
&= \left\{ \left(\frac{\partial \mathbb{X}}{\partial x_1} \vec{e}_1 + \frac{\partial \mathbb{X}}{\partial x_2} \vec{e}_2 + \frac{\partial \mathbb{X}}{\partial x_3} \vec{e}_3 \right) \left[ct + i\left(x_1\, i\, \vec{e}_1 + x_2\, j\, \vec{e}_2 + x_3\, k\, \vec{e}_3 \right) \right] \right\} \frac{\partial}{\partial \mathbb{X}} \\
&= \left\{ \vec{i} \cdot \left(\frac{\partial \mathbb{X}}{\partial x_1} dx_1 + \frac{\partial \mathbb{X}}{\partial x_2} dx_2 + \frac{\partial \mathbb{X}}{\partial x_3} dx_3 \right) \right\} \frac{\partial}{\partial \mathbb{X}} \\
&= \left\{ \vec{i} \cdot \left(i dx_1 + i dx_2 + i dx_3 \right) \right\} \left[\frac{1}{c} \frac{\partial}{\partial t} + \vec{i} \cdot i \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) \right] \\
&= \vec{i} \cdot d\vec{x} \frac{i}{c} \frac{\partial}{\partial t} + \left(\vec{i} \cdot i d\vec{x} \right)\left(\vec{i} \cdot i\vec{\nabla} \right) \\
&= d\vec{x} \cdot \vec{\nabla} + \vec{i} \cdot \left(\frac{i}{c} d\vec{x} \frac{\partial}{\partial t} - d\vec{x} \times \vec{\nabla} \right)
\end{aligned}$$

and finally

$$\frac{\partial \mathbb{X}}{\partial \vec{\mathbf{x}}} \frac{\partial}{\partial \mathbb{X}} d\vec{\mathbf{x}} \quad = d\vec{\mathbf{x}} \cdot \vec{\nabla} + \vec{i} \cdot \left(\frac{i}{c} d\vec{\mathbf{x}} \frac{\partial}{\partial t} - d\vec{\mathbf{x}} \times \vec{\nabla} \right) \quad \text{(A.4)}$$

The addition of (A.3) and (A.4) and the afterwards division by dt gives the total time differential in bi-quaternion form:

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} \right) + \vec{i} \cdot \left[i \left(c\vec{\nabla} + \frac{\vec{\mathbf{v}}}{c} \circ \frac{\partial}{\partial t} \right) - \left(\vec{\mathbf{v}} \times \vec{\nabla} \right) \right] \quad \text{(A.5)}$$

The special multiplication symbol ∘ indicates, that on applying this operator the scalar multiplication must be used for the scalar part and the cross product must be used for the vector part.

Appendix B

We start with the transformation:

$$\mathbb{X}' = \frac{\gamma}{c}\mathbb{U}^{*}\mathbb{X} = \gamma\left(1-\vec{i}\cdot \mathrm{i}\frac{\vec{v}}{c}\right)\left(ct+\vec{i}\cdot \mathrm{i}\vec{x}\right) \tag{B.1}$$

and derive

$$\left(ct'+\vec{i}\cdot \mathrm{i}\vec{x}'\right) = \gamma\left[\left(ct-\frac{\vec{v}\cdot\vec{x}}{c}\right)+\vec{i}\cdot c\left(\mathrm{i}\vec{x}-\mathrm{i}\vec{v}t-\frac{\vec{v}\times\vec{x}}{c}\right)\right] \tag{B.2}$$

With comparison of coefficients we find:

$$ct' = \gamma\left(ct-\frac{\vec{v}\cdot\vec{x}}{c}\right) \tag{B.3}$$

$$\vec{x}' = \gamma\left(\vec{x}-\vec{v}t\right) \tag{B.4}$$

The first equation (B.3) is well-known and corresponds to Lorentz transformation of time in vector form. The following derivation shows, that this is also valid for equation (B.4):

$$\begin{aligned}\vec{v}\cdot\gamma\left[\vec{x}-\vec{v}t\right] &= \gamma\vec{v}\cdot\vec{x}-\gamma v^2 t\\ &= \left(\gamma-1\right)\vec{v}\cdot\vec{x}+\vec{v}\cdot\vec{x}-\gamma v^2 t\\ &= \vec{v}\cdot\left[\vec{x}-\gamma\vec{v}t+\left(\gamma-1\right)\frac{\vec{v}}{v}\frac{\vec{v}\cdot\vec{x}}{v}\right]\end{aligned}$$

In many text-books the last term in square brackets is described as Lorentz transformation of position vector $\vec{x}$. By comparison of both terms in square brackets we find the identity

$$\gamma\left(\vec{x}-\vec{v}t\right) = \vec{x}-\gamma\vec{v}t+\left(\gamma-1\right)\frac{\vec{v}}{v}\frac{\vec{v}\cdot\vec{x}}{v} \tag{B.5}$$

This proves, that (B.4) is indeed the Lorentz transformation of $\vec{x}$. If $\vec{v}$ is collinear to $\vec{x}$, both equations reduce to the well-known one-dimensional transformation equations

$$t' = \gamma\left(t-\frac{vx}{c^2}\right) \tag{B.6}$$

$$x' = \gamma\left(x-vt\right) \tag{B.7}$$

Appendix C

We start with

$$\frac{1}{\mu}(\Delta\mathbb{A})\nabla\mathbb{A} = \mathbb{J}\nabla\mathbb{A} \quad \Rightarrow \quad -(\Delta\mathbb{A})\nabla\mathbb{A} - \mu\underline{\mathbb{F}} = 0 \tag{C.1}$$

Therein we have the terms:

$$-\nabla\mathbb{A} = \frac{1}{c}\mathbb{E} = s + \vec{i}\cdot\left(\frac{i}{c}\vec{E}+\vec{B}\right) \tag{C.2}$$

$$\Delta\mathbb{A} = \frac{1}{c}\left(\vec{\nabla}\cdot\vec{E} - \frac{\partial s}{\partial t}\right) - i\vec{\nabla}\cdot\vec{B} + \vec{i}\cdot\left[i\left(\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}s\right) - \frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)\right] \tag{C.3}$$

$$-\mu\underline{\mathbb{F}} = -\mu\rho\left\{\frac{1}{c}\left(\vec{v}\cdot\vec{E} + c^2 s\right) - i\vec{v}\cdot\vec{B} + \vec{i}\cdot\left[i\vec{v}\times\vec{B} + i\vec{E} + i\vec{v}s + \frac{1}{c}\left(c^2\vec{B} - \vec{v}\times\vec{E}\right)\right]\right\} \tag{C.4}$$

Equations (C.4) is already the second term in (C.1). Therefore we need to calculate the first term only:

$$-(\Delta\mathbb{A})\nabla\mathbb{A} = \left\{\begin{matrix} \frac{1}{c}\left(\vec{\nabla}\cdot\vec{E} - \frac{\partial s}{\partial t}\right) - i\vec{\nabla}\cdot\vec{B} + \\ \vec{i}\cdot\left[i\left(\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}s\right) - \frac{1}{c}\left(\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)\right] \end{matrix}\right\}\left\{\begin{matrix} s + \\ \vec{i}\cdot\left(\frac{i}{c}\vec{E}+\vec{B}\right) \end{matrix}\right\} \tag{C.5}$$

Imaginary scalar term

We find for the imaginary scalar term of (C.5):

$$-s\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0} - \underbrace{\vec{B}\cdot\left(\vec{\nabla}\times\vec{B}\right)}_{0} + \frac{1}{c^2}\underbrace{\vec{E}\cdot\left(\vec{\nabla}\times\vec{E}\right)}_{0} + \underbrace{\frac{1}{c^2}\vec{E}\cdot\underbrace{\frac{\partial\vec{B}}{\partial t}}_{=-\vec{\nabla}\times\vec{E}}}_{0} + \vec{B}\cdot\underbrace{\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}_{=\vec{\nabla}\times\vec{B}+\vec{\nabla}s} - \vec{B}\cdot\vec{\nabla}s = \underbrace{\vec{B}\cdot\left(\vec{\nabla}\times\vec{B}\right)}_{0} = 0$$

Together with the imaginary scalar term of (C.4) we get: **-**$\mu\rho$**v·B** = **-**μ**J·B** in (C.1):

$$\mu\vec{J}\cdot\vec{B} = 0 \tag{C.6}$$

Real scalar term

Calculating the real scalar term of (C.5) we find

$$\frac{1}{c}\left\{s\left(\vec{\nabla}\cdot\vec{E}\right) - s\frac{\partial s}{\partial t} + \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \frac{1}{c^2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{E}\cdot\vec{\nabla}s + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)\right\} \tag{C.7}$$

And expanded:

$$\frac{1}{c}\left\{\begin{matrix} s\left(\vec{\nabla}\cdot\vec{E}\right) - s\frac{\partial s}{\partial t} + \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \frac{1}{c^2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{E}\cdot\vec{\nabla}s + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) \\ + \quad 2\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - 2\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) \end{matrix}\right\} \tag{C.8}$$

Using FARADAY's law

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu\vec{J} - \vec{\nabla}s$$

we get

$$\frac{1}{c}\left\{\begin{array}{l} s\left(\vec{\nabla}\cdot\vec{E}\right)-s\dfrac{\partial s}{\partial t}+\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)-\dfrac{1}{c^2}\vec{E}\cdot\dfrac{\partial\vec{E}}{\partial t}+\vec{E}\cdot\vec{\nabla}s+\vec{B}\cdot\dfrac{\partial\vec{B}}{\partial t}+\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) \\ +\quad \dfrac{2}{c^2}\vec{E}\cdot\dfrac{\partial\vec{E}}{\partial t}+2\mu\vec{E}\cdot\vec{J}-2\vec{E}\cdot\vec{\nabla}s-2\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) \end{array}\right\} \tag{C.9}$$

or

$$\frac{1}{c}\left\{\frac{1}{c^2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}+\underbrace{\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right)-\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)}_{\nabla\cdot\left(\vec{E}\times\vec{B}\right)}+2\mu\vec{E}\cdot\vec{J}-\vec{E}\cdot\vec{\nabla}s+s\left(\vec{\nabla}\cdot\vec{E}\right)-s\frac{\partial s}{\partial t}\right\}$$

Then, with

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon}+\frac{\partial s}{\partial t}$$

follows

$$\frac{1}{c}\left\{\frac{1}{c^2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)+2\mu\vec{E}\cdot\vec{J}-\vec{E}\cdot\vec{\nabla}s+s\frac{\rho}{\varepsilon}\right\} \tag{C.10}$$

Inserting the real scalar part of (C.4):

$$\frac{1}{c}\left\{\mu\rho\left(\vec{v}\cdot\vec{E}+c^2 s\right)\right\}=\frac{i}{c}\left\{\mu\vec{J}\cdot\vec{E}+s\frac{\rho}{\varepsilon}\right\}$$

and (C.9) in (C.1), we get finally Poynting's theorem in different notations

$$\frac{1}{c^2}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)+\vec{E}\cdot\left(\mu\vec{J}-\vec{\nabla}s\right)=0 \tag{C.11}$$

$$\frac{1}{2}\left(\frac{1}{c^2}\frac{\partial E^2}{\partial t}+\frac{\partial B^2}{\partial t}\right)+\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)+\vec{E}\cdot\left(\mu\vec{J}-\vec{\nabla}s\right)=0 \tag{C.12}$$

$$\frac{1}{2}\left(\frac{\partial\varepsilon E^2}{\partial t}+\frac{\partial}{\partial t}\frac{B^2}{\mu}\right)+\vec{\nabla}\cdot\left(\vec{E}\times\frac{\vec{B}}{\mu}\right)+\vec{E}\cdot\left(\vec{J}-\frac{1}{\mu}\vec{\nabla}s\right)=0 \tag{C.13}$$

$$\frac{1}{2}\left(\varepsilon\frac{\partial E^2}{\partial t}+\mu\frac{\partial H^2}{\partial t}\right)+\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)+\vec{E}\cdot\left(\vec{J}-\frac{1}{\mu}\vec{\nabla}s\right)=0 \tag{C.14}$$

Inserting Maxwell's equation

$$0=s\left(\vec{\nabla}\cdot\vec{E}-\frac{\partial s}{\partial t}-\frac{\rho}{\varepsilon}\right)=\vec{\nabla}\cdot\left(s\vec{E}\right)-\left(\vec{\nabla}s\right)\cdot\vec{E}-s\frac{\partial s}{\partial t}-s\frac{\rho}{\varepsilon}$$

in (C.12), we further find

$$\frac{1}{2}\left(\frac{\partial s^2}{\partial t}+\frac{1}{c^2}\frac{\partial E^2}{\partial t}+\frac{\partial B^2}{\partial t}\right)+\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}-s\vec{E}\right)+\vec{E}\cdot\vec{J}+s\frac{\rho}{\varepsilon}=0 \tag{C.15}$$

Imaginary vector term

We find for the imaginary vector term of (C.5):

$$-\vec{B}\left(\vec{\nabla}\cdot\vec{B}\right)+s\left(\vec{\nabla}\times\vec{B}\right)-\frac{s}{c^2}\frac{\partial\vec{E}}{\partial t}+s\,\vec{\nabla}\,s+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$
$$+\vec{\nabla}\,s\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{B}}{\partial t}\times\vec{E}-\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right) \tag{C.16}$$

We take the bi-quaternion MAXWELL equations

$$\vec{\nabla}\cdot\vec{B}=0 \quad \text{und} \quad \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\mu\vec{J}-\vec{\nabla}\,s$$

to reduce (C.16) to

$$s\mu\vec{J}+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\times\vec{B}+\vec{\nabla}\,s\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{B}}{\partial t}\times\vec{E}-\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)$$

or

$$\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}-\frac{1}{c^2}\left[\frac{\partial E}{\partial t}\times\vec{B}+\frac{\partial B}{\partial t}\times\vec{E}\right]-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\vec{\nabla}\,s\times\vec{B}+s\mu\vec{J}$$

To change the sign of third term, we add the last term

$$\underbrace{\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}+\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}}_{\frac{1}{c^2}\nabla\mathbf{T}}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{B}}{\partial t}\times\vec{E}-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\vec{\nabla}\,s\times\vec{B}+s\mu\vec{J}-\frac{2}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}$$

use AMPERE's law $\qquad \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$

and get

$$\underbrace{\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}+\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}}_{\frac{1}{c^2}\nabla\mathbf{T}}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\times\vec{B}+\frac{1}{c^2}\frac{\partial\vec{B}}{\partial t}\times\vec{E}-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\vec{\nabla}\,s\times\vec{B}+s\mu\vec{J}$$

Therof we have

$$\underbrace{\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}+\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}}_{\frac{1}{c^2}\vec{\nabla}\mathbf{T}}-\frac{1}{c^2}\underbrace{\left[\frac{\partial\vec{E}}{\partial t}\times\vec{B}+\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right]}_{\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)}-\frac{1}{c^2}\frac{\partial s}{\partial t}\vec{E}+\vec{\nabla}\,s\times\vec{B}+s\mu\vec{J} \tag{C.17}$$

With two vector identities and with the FARADAY equation, multiplied by *s*

$$\frac{\partial s}{\partial t}\vec{E}=\frac{\partial}{\partial t}\left(s\,\vec{E}\right)-s\frac{\partial\vec{E}}{\partial t},\;\; \left(\vec{\nabla}\,s\right)\times\vec{B}=\vec{\nabla}\times\left(s\,\vec{B}\right)-s\left(\vec{\nabla}\times\vec{B}\right),\;\; 0=s\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}-\vec{\nabla}\times\vec{B}+\mu\vec{J}\right)$$

equations (C.17) becomes

$$\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}+\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}+s\,\vec{E}\right)+\vec{\nabla}\times s\,\vec{B} \tag{C.18}$$

Inserting (C.18) and the imaginary vector term of (C.4)

$$\mu\rho\,\vec{i}\cdot\left[\left(\vec{E}+\vec{v}\times\vec{B}\right)+\vec{v}\,s\right]=\mu\,\vec{i}\cdot\left[\rho\vec{E}+\vec{J}\times\vec{B}+\vec{J}\,s\right]$$

into (C.1), we finally get

$$\frac{1}{c^2}\vec{E}\left(\vec{\nabla}\cdot\vec{E}\right)+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}+\frac{1}{c^2}\left(\vec{\nabla}\times\vec{E}\right)\times\vec{E}$$
$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}+s\,\vec{E}\right)+\vec{\nabla}\times s\,\vec{B}=\mu\rho\vec{E}+\mu\vec{J}\times\vec{B}+\mu\,s\,\vec{J} \tag{C.19}$$

Appendix D: Quantum mechanics

Relativistic wave equation

In 1937 CONWAY[3] has shown a possible notation for the relativistic wave equation, if the Hamilton'ian units are used as pre- and post factors in his formulas. In this section we derive the relativistic wave equation with the bi-quaternion notation used already in previous chapters. We start with the momentum law:

$$\mathbb{p}_m = m\mathbb{V} \tag{4.1}$$

and with the definition of total energy of a mass

$$E \equiv c\left|\mathbb{p}\right| \tag{4.2}$$

we can derive EINSTEIN's formula by using (1.24) with k = 1..3:

$$E^2 = m^2c^4 + c^2\sum_k p_k^2 \tag{4.3}$$

Until this point we have used a scalar value representing the energy, which for example takes a constant value for a resting body. But quantum physics (experiments) have shown, that this is not quite correct but that energy is merely an oscillating phenomenon and therefore must satisfy a wave equation. Therefore equation (4.3) can be seen as a ‚static average' equation of a collection of many „energy oscillators". But for a single particle the oscillatory behavior of its energy can clearly be observed.

Equation (4.3) is already in a quadratic form as this is the case also for the differential operators of a wave equation. To find the wave equation behind the energy equation, we use the established substitutions for the differentials as known in quantum mechanics:

$$E \to -\frac{\hbar}{ic}\frac{\partial}{\partial t} \qquad \text{und} \qquad p_k \to \frac{\hbar}{i}\frac{\partial}{\partial x_k} \ , \tag{4.4)a, b}$$

where $\hbar$ is PLANCK's constant divided by 2π. Now this differentials must be applied to a new, unknown function. This is nothing else than the (dimensionless) wave function $\boldsymbol{\Psi}$. This wave function is again a bi-quaternion:

$$\boldsymbol{\Psi} = \psi_0 + \vec{i}\cdot i\psi_k \tag{4.5}$$

By inserting (4.5) into (4.3) we get directly DIRAC's relativistic wave equation:

$$\frac{1}{c^2}\frac{\partial\boldsymbol{\Psi}}{\partial t^2} - \vec{\Delta}\boldsymbol{\Psi} + \frac{m^2c^2}{\hbar^2}\boldsymbol{\Psi} = 0 \tag{4.6}$$

Particle without external potential fields

DIRAC has solved the energy equation (4.3)

$$E = \pm c\sqrt{m^2c^2 + \sum_k p_k^2} \tag{4.7}$$

by taking the square root with the introduction of 4x4 matrixes. On the other side equation (4.2) offers an other possible derivation without taking a square root, if directly taking the quaternion momentum. But the sign of the quaternion momentum should not change. This can be achieved with a modification of equation (4.2):

$$\mathbb{E} \equiv \pm c\mathbb{p} \qquad \text{then it is} \quad E = \left|\mathbb{E}\right| = \pm c\left|\mathbb{p}\right| \tag{4.8}$$

The two possible sign for the energy state in (4.7) and (4.8) has motivated DIRAC to postulate the existence of anti-particles – especially the positron. By inserting of the substitutions (4.4) into (4.8) another DIRAC equation ca be obtained:

$$\hbar\left(\frac{\partial\boldsymbol{\Psi}}{\partial x_1}i+\frac{\partial\boldsymbol{\Psi}}{\partial x_2}j+\frac{\partial\boldsymbol{\Psi}}{\partial x_3}k+\frac{1}{c}\frac{\partial\boldsymbol{\Psi}}{\partial t}\right)-mc\boldsymbol{\Psi}=0\ , \tag{4.9}$$

On a first glance this equation differs form the original DIRAC equation because no matrixes are used. But the HAMILTON'ian units can also be written as matrixes (see Appendix E). The multiplication of (4.9) with this HAMILTON'ian matrixes gives the following equation system:

$$\begin{aligned}
&\frac{\hbar}{c}\frac{\partial\psi_0}{\partial t}-mc\psi_0-\frac{\hbar}{i}\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}+\frac{\partial\psi_3}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_1}{\partial t}-mc\psi_1+\hbar\left(i\frac{\partial\psi_0}{\partial x_1}-\frac{\partial\psi_3}{\partial x_2}+\frac{\partial\psi_2}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_2}{\partial t}-mc\psi_2+\hbar\left(\frac{\partial\psi_3}{\partial x_1}+i\frac{\partial\psi_0}{\partial x_2}-\frac{\partial\psi_1}{\partial x_3}\right)=0\\
&\frac{\hbar}{c}\frac{\partial\psi_3}{\partial t}-mc\psi_3-\hbar\left(\frac{\partial\psi_2}{\partial x_1}-\frac{\partial\psi_1}{\partial x_2}-i\frac{\partial\psi_0}{\partial x_3}\right)=0
\end{aligned}\ , \tag{4.10}$$

This system contains four equations for a particle without external fields as originally proposed by DIRAC. Now, in an analogue way the equation for a particle within external fields can be described.

Particle within external potential field

The momentum on a charged particle changes with an external potential field. If the momentum of an external field on a charged particle q is defined as

$$\mathbb{p}_q=-q\mathbb{A} \tag{4.11}$$

then it follows for its energy

$$E_q\equiv c\left|\mathbb{p}_q\right|=-cq\left|\mathbb{A}\right| \tag{4.12}$$

and

$$\mathbb{E}_q=\mp cq\mathbb{A} \tag{4.13}$$

The total energy is then

$$\mathbb{E}=c\left(\pm\mathbb{p}_q\mp q\mathbb{A}\right) \tag{4.14}$$

Again the extended DIRAC equation follows with the substitution of energy and momentum to:

$$\hbar\left\{\left(\frac{\partial}{\partial x_1}-qA_1\right)i+\left(\frac{\partial}{\partial x_2}-qA_2\right)j+\left(\frac{\partial}{\partial x_3}-qA_3\right)k+\frac{1}{c}\frac{\partial}{\partial t}\right\}\boldsymbol{\Psi}-\left(mc+q\frac{\varphi}{c}\right)\boldsymbol{\Psi}=0 \tag{4.15}$$

Again quaternions can be used instead of matrixes.

Appendix E: Matrices in quaternion form

According Arthur CAYLEY, complex numbers can be expressed with matrices:

$$a + ib = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with} \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.16}$$

Example:

$$i^2 = ii = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \tag{4.17}$$

The HAMILTON'ian units of a quaternion build together with the numbers 1 and –1 a non ABEL'ian group of eighth order. Its first four positive elements are:

$$\mathit{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathit{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathit{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.18}$$

Replacing the imaginary unit i with the corresponding matrix (4.16) gives:

$$\mathit{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \mathit{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathit{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.19}$$

The square of this matrixes always return the value –1 as requested by the definition of the HAMILTON'ian unit vectors (1.2).