The Total Time Derivative with Bi-Quaternion Electrodynamics

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In a recent paper [3] we have shown that the basic equations of electrodynamics can be cast into a bi-quaternion form. In this paper I present an other general way how the set of four equations of the generalised Lorentz force can be derived by introducing a new operator: The bi-quaternion total time derivative operator.

Introduction

Bi-quaternions are very useful for a compact description of electrodynamics. A bi-quaternion is defined as

$$\mathrm{X} = \mathrm{x}_0 + \mathrm{i}\mathrm{y}_0 + \vec{\boldsymbol{i}} \cdot \left(\vec{\mathbf{x}} + \mathrm{i}\vec{\mathbf{y}}\right) \tag{1}$$

where

$$\mathrm{i} = \sqrt{-1} \quad \text{and} \quad \vec{\boldsymbol{i}} \cdot \vec{\mathbf{x}} = i\,\mathrm{x}_1 + j\,\mathrm{x}_2 + k\,\mathrm{x}_3 \tag{2}$$

and

$$i^2 = j^2 = k^2 = ijk = -1 \tag{3}$$

and

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ij = -ji & jk = -kj & ki = -ik\,. \end{array}$$

Then the four dimensional position bi-quaternion is

$$\mathrm{X} = \mathrm{ict} + \vec{\boldsymbol{i}} \cdot \vec{\mathbf{x}} \tag{4}$$

the four dimensional bi-quaternion velocity is

$$\mathrm{V} = \mathrm{ic} + \vec{\boldsymbol{i}} \cdot \vec{\mathbf{v}} \tag{5}$$

the four dimensional bi-quaternion electromagnetic potential is

$$\mathrm{A} = \frac{\mathrm{i}}{\mathrm{c}}\Phi + \vec{\boldsymbol{i}} \cdot \vec{\mathbf{A}} \tag{6}$$

the four dimensional bi-quaternion current density is

$$\mathrm{J} = \rho\mathrm{V} = \mathrm{ic}\rho + \vec{\boldsymbol{i}} \cdot \rho\vec{\mathbf{v}} \tag{7}$$

and finally the four dimensional bi-quaternion force density is

$$\mathrm{F} = \frac{\mathrm{i}}{\mathrm{c}}\mathrm{P} + \vec{\boldsymbol{i}} \cdot \vec{\mathbf{F}} \tag{8}$$

Then the bi-quaternion spatial differential operators are defined as follows:

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Nabla : $\nabla = \frac{i}{c}\frac{\partial}{\partial t} + \vec{\boldsymbol{i}} \cdot \vec{\nabla} \qquad \vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ (9)

d'Alembert : $\Box = \frac{i}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}$ (10)

With this definitions the generalized Lorentz force is simply expressed as [3]:

$$F = J\nabla A \tag{11}$$

It shall now be shown, that this force density can be derived form the total time derivative of the bi-quaternion potential also.

The total time derivative bi-quaternion operator

Lets have a bi-quaternion field A = A(X) where X is in turn a function of time and space, so that $A = A(X(t, x_1, x_2, x_3))$. Then the total derivative of A becomes

$$dA = \frac{\partial A(X)}{\partial X} dX = \frac{\partial A}{\partial X}\frac{\partial X}{\partial t} dt + \frac{\partial A}{\partial X}\frac{\partial X}{\partial \vec{\mathbf{x}}} d\vec{\mathbf{x}} = -\left(\frac{\partial X}{\partial t}\frac{\partial A}{\partial X} dt + \frac{\partial X}{\partial \vec{\mathbf{x}}}\frac{\partial A}{\partial X} d\vec{\mathbf{x}}\right) \tag{12}$$

where

$$\frac{\partial}{\partial X} = \nabla \quad \text{and} \quad \frac{\partial}{\partial \vec{\mathbf{x}}} = \vec{\nabla} \tag{13}$$

The operator of the first term in (12) is then expanded to

$$\frac{\partial X}{\partial t}\frac{\partial}{\partial X} dt = \left(-\frac{\partial}{\partial t} + ic\,\vec{i}\cdot\vec{\nabla}\right) dt \tag{14}$$

The operator of the second term in (12) has the following expansion:

$$\begin{aligned}
\frac{\partial X}{\partial \vec{\mathbf{x}}}\frac{\partial}{\partial X} d\vec{\mathbf{x}} &= \left(\frac{\partial X}{\partial x_1} dx_1 + \frac{\partial X}{\partial x_2} dx_2 + \frac{\partial X}{\partial x_3} dx_3\right)\frac{\partial}{\partial X} \\
&= \left(i\,dx_1 + j\,dx_2 + k\,dx_3\right)\left(\frac{i}{c}\frac{\partial}{\partial t} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right) \\
&= \vec{i}\cdot d\vec{\mathbf{x}}\frac{i}{c}\frac{\partial}{\partial t} + \begin{pmatrix} i\,dx_1 \\ j\,dx_2 \\ k\,dx_3 \end{pmatrix}\left(i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right) \\
&= \vec{i}\cdot\left(\frac{i}{c} d\vec{\mathbf{x}}\frac{\partial}{\partial t} + d\vec{\mathbf{x}}\times\vec{\nabla}\right) - d\vec{\mathbf{x}}\cdot\vec{\nabla}
\end{aligned} \tag{15}$$

Adding (14) and (15) together and divide by dt results in the bi-quaternion total time derivative:

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\nabla}\right) - \vec{i}\cdot\left[\left(\vec{\mathbf{v}}\times\vec{\nabla}\right) + i\left(c\vec{\nabla} + \frac{\vec{\mathbf{v}}}{c}\frac{\partial}{\partial t}\right)\right] \tag{16}$$

Surprisingly the multiplication of bi-quaternion velocity with the Nabla operator yields the same result, but with negative sign:

$$V\nabla = -\left(\frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla}\right) + \vec{\boldsymbol{i}} \cdot \left[\left(\vec{\mathbf{v}} \times \vec{\nabla}\right) + i\left(c\vec{\nabla} + \frac{\vec{\mathbf{v}}}{c}\frac{\partial}{\partial t}\right)\right] \tag{17}$$

As the result, the bi-quaternion total time derivative can be written very compact as

$$\boxed{\frac{d}{dt} = -V\nabla} \tag{18}$$

It is now easy to proof that the bi-quaternion force density can also be obtained by using the total time derivative:

$$F = J\nabla A = \rho V\nabla A = -\rho\frac{dA}{dt} \tag{19}$$

Expanding equation (19) gives four equations:

$$\begin{aligned}
\rho\vec{\mathbf{v}} \cdot \left(\vec{\nabla} \times \vec{\mathbf{A}}\right) &= 0 \\
-\rho\left[\vec{\mathbf{v}} \cdot \left(\frac{\partial\vec{\mathbf{A}}}{\partial t} + \vec{\nabla}\Phi\right) + c^2\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla} \cdot \vec{\mathbf{A}}\right)\right] &= P \\
-\rho\left[\left(\frac{\partial\vec{\mathbf{A}}}{\partial t} + \vec{\nabla}\Phi\right) - \vec{\mathbf{v}} \times \left(\vec{\nabla} \times \vec{\mathbf{A}}\right) + \vec{\mathbf{v}}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla} \cdot \vec{\mathbf{A}}\right)\right] &= \vec{\mathbf{F}} \\
\rho\left[\left(\vec{\nabla} \times \vec{\mathbf{A}}\right) + \frac{\vec{\mathbf{v}}}{c^2} \times \left(\frac{\partial\vec{\mathbf{A}}}{\partial t} + \vec{\nabla}\Phi\right)\right] &= 0
\end{aligned} \tag{20}$$

A closer look to the new proposed total time derivative (16) shows some familiar terms, as for example the well known scalar total time derivative operator. Recently some proposals for a total time derivative of the vector potential **A** has been published by Wesley [4] and Phipps [2] and also ten years before by Mocanu [1]. All this different but similar total time derivatives are based on a separation of spatial and temporal dimensions, i.e. they are not derived consequently from a four dimensional topology. From equation (20) we can extract two equations for the total time derivative of the vector potential:

$$\frac{d\vec{\mathbf{A}}}{dt} = \frac{\partial\vec{\mathbf{A}}}{\partial t} - \vec{\mathbf{v}} \times \left(\vec{\nabla} \times \vec{\mathbf{A}}\right) + \vec{\mathbf{v}}\left(\vec{\nabla} \cdot \vec{\mathbf{A}}\right) \tag{21}$$

or

$$\frac{d\vec{\mathbf{A}}}{dt} = \frac{\partial\vec{\mathbf{A}}}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla}\vec{\mathbf{A}} - \left(\vec{\mathbf{v}} \times \vec{\nabla}\right) \times \vec{\mathbf{A}}\ . \tag{22}$$

References

[1] Mocanu C.I., "Hertzian Relativistic Electrodynamics and its Associated Mechanics", *Hadronic Press*, Palm Harbor, FL , **1** (1991) 34-38

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