Analysing Transformers in the Magnetic Domain

©Cyril Smith, February 2006

1. Introduction

The electrical equivalent circuit for a transformer has served electrical engineers well for over a century, but it has some disadvantages. Essentially it models the transformer action by presuming a "perfect transformer" which has no losses and obeys certain rules i.e. (a) the AC voltages between coils are in the ratio of their turns-ratio with zero phase shift, (b) the AC current between coils are in the inverse ration of the turns ratio, again with zero phase shift which lead to (c) the load resistance reflected from secondary coil to primary coil is in the inverse ratio of the square of the turns ratio. Around this *perfect* device are placed electrical components which represent the imperfections of a *real* device. Often the magnitudes of some of these imperfections are found empirically. Whilst this method is satisfactory for most transformer applications, it does not deal with transformers which have unusual features which affect their magnetic characteristics. For instance the presence of a permanent magnet as part of the magnetic circuit, and its effect on the electrical characteristics, cannot be determined. Also complex arrangements where multiple magnetic paths and coils are involved require special attention which involves some magnetic domain solutions in order to determine the parameters of the equivalent electrical circuit, but these magnetic domain analyses go no further than constructing a network of reluctances. Time delays or phase shifts which result from magnetic parameters are another complication which require further ad-hoc add-ons to the equivalent circuit. Reactive loads are another situation which is not readily addressed, and while the equivalent circuit attempts to model the electrical effects by reflecting the load reactance through the "perfect" transformer, it does not account for the changed magnetic flux and its interaction with the core material or other magnetic components.

Of particular note is the fact that the primary and secondary load currents may be several orders of magnitude larger than the magnetizing current, so each by itself has the capability of creating magnetic flux of great magnitude far in excess of that which the core material can accommodate. That they do not generally do so is because they oppose each other with respect to flux generation, that opposition and perfect flux cancellation being a hidden assumption implicit in the equivalent circuit. In practice such flux cancellation can only be considered near perfect if primary and secondary are bifilar wound, which is not generally the case. If primary and secondary coils are wound on different parts of the magnetic circuit, the load currents *do* create non-circulatory flux which is driven outside the core, that flux being either ignored or accounted for only in an ad-hoc manner. (The series inductance which is intended to account for leakage *magnetization* flux, i.e. the portion of quadrature flux created at the primary coil which does not reach the secondary, is normally determined by a measurement of primary inductance with the secondary short circuited. That determination takes no account of the imperfection at the secondary where its coil resistance prevents it from being an absolute short circuit.)

This report describes a method for the dynamic analysis of transformers in the *magnetic* domain. In a manner it is the inside-out-version of the electrical equivalent circuit. The magnetic "circuit" is modelled along well known lines where magnetic flux is treated like current and magnetic reluctance like resistance. However it goes beyond the simple "magnetic Ohm's Law" by introducing other magnetically reactive components which represent the outside electrical world. This allows a time or frequency domain analysis to be carried out on the magnetic flux, from which the transformer electrical characteristics automatically follow. Such a procedure gives a deeper insight into the magnetic behaviour, and enables more accurate modelling of unusual systems or those working outside their normal envelope.

2. Magnetic Ohm's Law

This feature will be found in any good treatise on magnetism, and is repeated here for completeness and as a starting point. Magnetic reluctance $\mathbb{R}$ is treated like electrical resistance R, magnetic flux Φ like current I and mmf U like voltage V. Hence the mmf drop across a length of core material of reluctance $\mathbb{R}$ is given by $U=\Phi\mathbb{R}$, which is the magnetic Ohm's Law equivalent of V=IR. And just as the resistance of a rod of resistive material is given by $R = {}^{l}/_{\sigma A}$ where l is the length, A the cross sectional area and σ the conductance, so the reluctance of a rod of ferromagnetic material is $\mathbb{R} = {}^{l}/_{\mu A}$ where μ is the absolute permeability. Treating sections of the magnetic circuit in this way enables the reluctance of a core made up of different cross sections to be determined. Also the distribution of flux among different branches of a complex core arrangement can be determined by solving as a network of reluctances.

For a core with a single coil the reluctance plays it part in the inductance as $L=N^2/\mathbb{R}$ where N is the number of turns. When the coil carries a current I its mmf is U=NI ampere turns, hence "Ohm's Law" gives the flux as $\Phi=NI/\mathbb{R}$. Thus flux $\Phi=LI/N$ (not to be confused with flux *linkage* which is often loosely called flux and given as $\Phi=LI$). Figure 1 shows a simple inductor and its "Ohms Law" equivalent circuit.

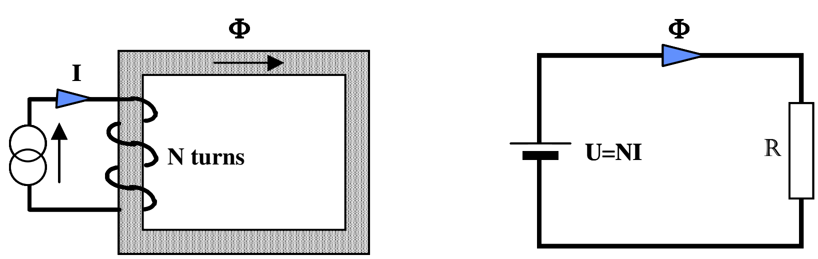


Figure 1. A simple Inductor and its Equivalent Circuit

Often a transformer will have an air gap so as to prevent the core material from going into saturation. This is easily accounted for by having two reluctances in series, Figure 2.

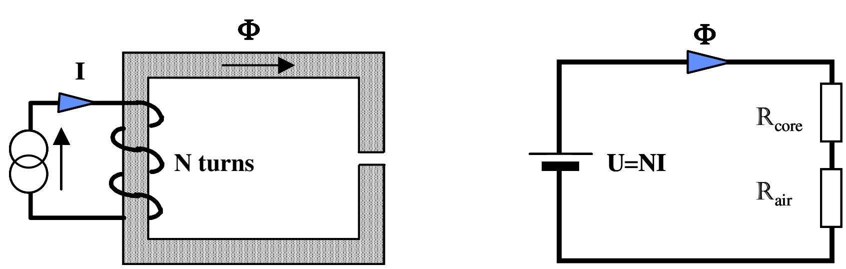


Figure 2. Core with Air Gap

An inductor which doesn't have a closed core magnetic path is a common component. For a solenoidal coil with no core there are standard formulae for calculating inductance, but difficulty occurs when the solenoid has a permeable core. This is usually dealt with by introducing the *demagnetizing* factor, which is dependent on the core geometry. Why there should be such a demagnetization can be easily seen by considering the air gap in Figure 2 made larger and larger until only a small portion of the core remains inside the coil.

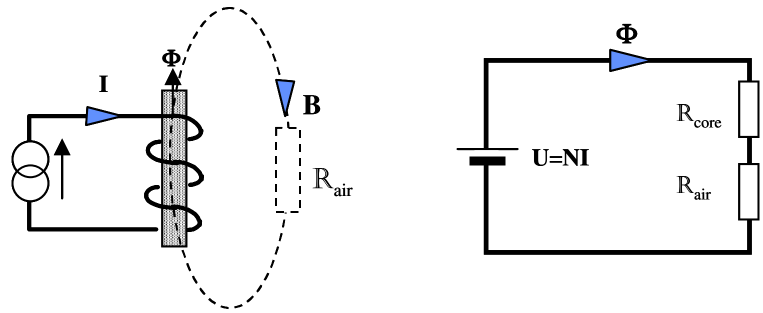


Figure 3. Solenoid

All of the flux now flows through the air forming the B field around the solenoid. This distribution of flux can still be represented by an air reluctance, and the demagnetizing factor is then seen simply as a means for calculating how the input mmf is distributed across the two reluctances. The mmf drop across the air reluctance reduces the mmf across the core (between the core faces) which appears as a reduced H value within the core, hence the term *demagnetization*. It will be seen later that representing flux lines through air by a reluctance $\mathbb{R}_{air}$ is also a feature of transformers.

3. Transformers

If we now turn to transformers with AC input, in addition to the primary coil of N_p turns the core has a secondary winding of N_s turns. The output voltage is $N_s d\Phi/dt$ which drives a current I_s through a load resistor R_{load} given by $I_s=(N_s/R_{load})d\Phi/dt$. The back mmf induced into the magnetic circuit is thus $U=-N_sI_s=-(N_s^2/R_{load})d\Phi/dt$. We can view the term in parentheses as a "magnetic inductance" $\mathbb{L}_S$ of value N_s^2/R_{load}, since $U=-\mathbb{L}_S d\Phi/dt$ has exactly the same form as $V=-LdI/dt$. The transformer is shown in Figure 4 with an input from a (high impedance) current source.

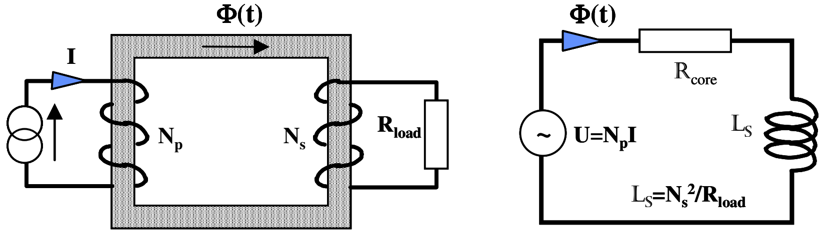


Figure 4. Current Driven Transformer

Most transformers are driven from low impedance voltage sources, and we wish to determine the transformer characteristics for that combination. To do this we have to take account of the effective source impedance which is generally the DC resistance of the primary coil, R_p. The back emf generated across the primary coil is V_b=-N_pdΦ/dt so, with an input voltage of V_{in} the voltage across R_p becomes V_{in}-V_b yielding an input current of (V_{in}-V_b)/R_p. The mmf applied to the core is thus N_p(V_{in}-V_b)/R_p which, with substitution for V_b, becomes U= N_p(V_{in}-N_pdΦ/dt)/R_p. This can be split into two mmf's, U=N_pV_{in}/R_p and U_1=-(N_p^2/R_p)dΦ/dt, the latter being of the same form as V=-LdI/dt. The primary mmf can therefore be modelled as an input U in series with a magnetic inductor $\mathbb{L}_P=N_p^2/R_p$.

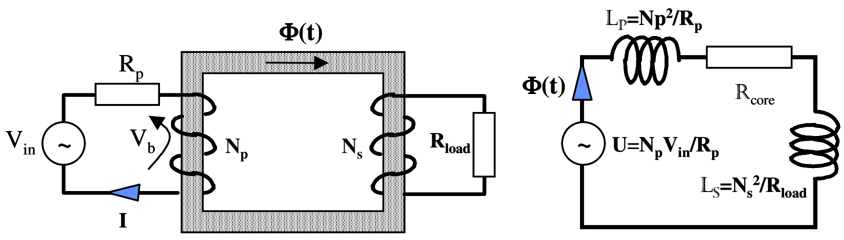


Figure 5. Voltage Driven Transformer

Figure 5 shows the voltage driven transformer with its equivalent circuit. Analysing the magnetic domain circuit is a trivial task since it is equivalent to only two inductors and a resistor in series. It correctly relates the phase and amplitude of the flux Φ to the input voltage V_{in} (U and V_{in} are in phase). From that analysis the electrical performance naturally arises as follows.

$$|\Phi| = \frac{U}{\sqrt{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2}} \qquad (1)$$

Phase angle $\phi = \tan - 1\left(\frac{\omega(\mathbb{L}_p + \mathbb{L}_s)}{\mathbb{R}_{core}}\right)$ (2)

Real and imaginary components of Φ are

$$\Phi = U\left(\frac{\mathbb{R}_{core}}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2} + j\frac{\omega(\mathbb{L}_s + \mathbb{L}_p)}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2}\right) \qquad (3)$$

Imaginary and real components of dΦ/dt are

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = j\omega\Phi = U\left(j\frac{\omega\mathbb{R}_{core}}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2} - \frac{\omega^2(\mathbb{L}_s + \mathbb{L}_p)}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2}\right) \qquad (4)$$

The secondary voltage V_s is then

$$V_s = UN_s\left(j\frac{\omega\mathbb{R}_{core}}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2} - \frac{\omega^2(\mathbb{L}_s + \mathbb{L}_p)}{\mathbb{R}_{core}^{\ 2} + \omega^2(\mathbb{L}_p + \mathbb{L}_s)^2}\right) \qquad (5)$$

The imaginary term shows that there is a phase shift between input voltage and the secondary voltage. This phase shift is negligible under normal operating conditions where $\omega(\mathbb{L}_p + \mathbb{L}_s) >> \mathbb{R}_{core}$ whence the first term becomes negligible and (5) simplifies to

$$V_s = \frac{-UN_s}{\mathbb{L}_p + \mathbb{L}_s} \qquad (6)$$

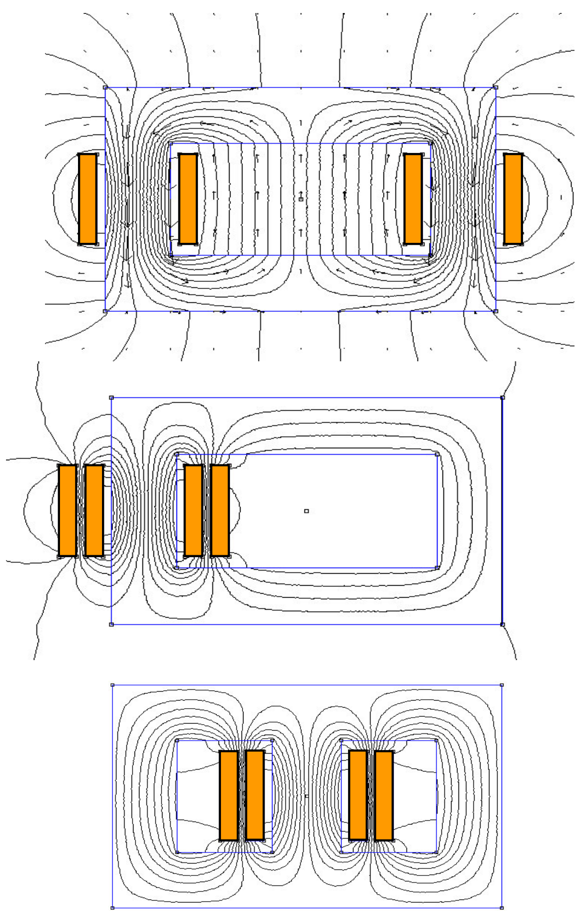
which with appropriate substitution for the magnetic terms becomes

$$V_s = \frac{-V_{in} N_p N_s}{N_p^{\ 2} + N_s^{\ 2} \dfrac{R_p}{R_{load}}} \qquad (7)$$

Since normally the load resistor R_{load} is much greater than the primary coil resistance R_p the second term in the denominator can be ignored so that $V_s = \frac{-V_{in} N_s}{N_p}$ which is the normal turns ratio formula with no phase shift. Similarly the primary coil back emf V_b, then the input current, can be determined using the same simplifications, the point here being that, when the transformer is operated outside its design regime those simplifications no longer apply. Surprisingly the trivial magnetic circuit in Figure 5 provides *all* the details necessary for the full analysis.

4. Flux outside the Core.

The above analysis takes no account of flux which escapes from the core. The following flux plots show the situation in normal transformers at the point where the magnetizing flux is passing through zero ($d\Phi/dt=0$) where the output and input voltages and load currents are at their peak values. These primary and secondary currents do not drive flux around the core, their ampere-turn mmf's being equal but in opposition, but it is clear that they do create flux within parts of the core.



(a)

(b)

(c)

Figure 6. Flux Patterns from Primary and Secondary Load Currents

When it is considered that load currents are several orders of magnitude larger than the magnetizing current, the flux lines in Figure 6 are not trivial. To take account of these the magnetic domain circuit must contain reluctance values representing the flux paths through the air. Figure 7 shows the addition of an air-reluctance, 7(a) corresponding with the transformer shown in 6(a) while 7(b) represents the transformers in 6(b) and 6(c).

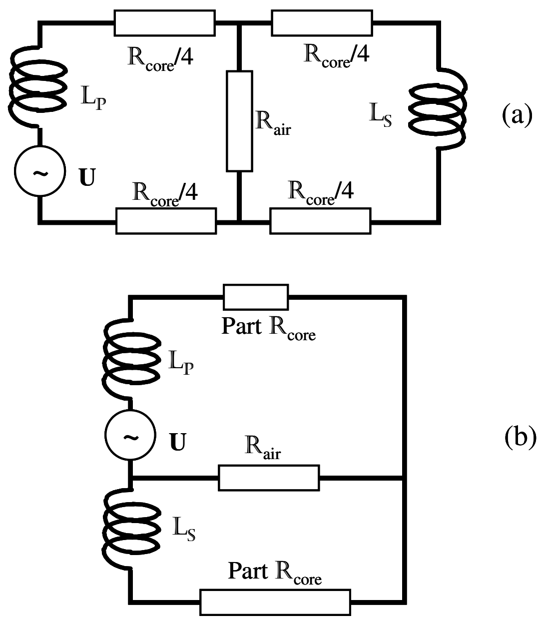


Figure 7. Showing the inclusion of Air Reluctance

It will be appreciated that the branch $\mathbb{R}_{air}$ now creates a T or π network giving the magnetic circuit a transfer impedance (under normal operating conditions this impedance is not noticed). Modelling in the magnetic domain allows this line-impedance to be seen for what it really is and how it may be affected by the incorporation of other magnetic components such as a permanent magnet or a variable reluctance.

5. Hysteresis Loss or Gain

It has been shown that a load taking power from the circuit, in this case a load resistor connected to the secondary coil, is modelled as a magnetic inductor $\mathbb{L}_S$. Hysteresis loss can similarly be modelled this way by the inclusion of an additional inductor $\mathbb{L}_H$. Its value may be determined by an actual loss measurement in the usual way, or by use of published figures for the *imaginary* component of core-material relative-permeability μ''. Representing permeability μ as a tensor $\mu=\mu' + j\mu''$ leads to the B v. H relationship (when driven sinusoidally) being an ellipse whose area represents energy density. The tensor μ is then a mathematical way of representing the classical BH loop which is traversed CCW and gives energy loss per cycle. This method can also include eddy current loss within that loop.

As an aside it may be noted that instability in magnetic materials found at microwave frequencies can be attributed to a *negative* value of μ''. The B v. H relationship is again an ellipse which is now traversed CW and represents energy *gain* per cycle. Should such a gain ever be manifest at the lower frequencies considered here it would be modelled as a *negative* value of $\mathbb{L}$. Recent researches into electron gases and semiconductors subjected to both a magnetic field and microwave radiation have exhibited similar instability described as *absolute negative resistance*. This negative resistance is at DC, so it obviously also applies to LF! It is possible that conduction electrons within magnetized ferromagnetic conductors could also exhibit instability gaining their energy, not from external microwave radiation but from internal radiation from individual electron Larmor precessions.

6. Reactive Loads.

The procedure used for reflecting an electrical load resistor R into the magnetic circuit to become a magnetic inductor $\mathbb{L}$ can similarly be used for reactive loads. This leads to an inductive load L becoming a magnetic reluctance given by $\mathbb{R}=N^2/L$.

A capacitive load is more difficult since it appears as an impedance unlike any observed in the electrical world. This new impedance acts somewhat like inductance, but whereas the back emf from an inductor is proportional to the *first* time-differential of the current, here the back emf is proportional to the *second* differential. The symbol D has been given to this impedance which obeys $V=-Dd^2I/dt^2$. In the electrical world such an impedance could be artificially created using an active circuit with appropriate feedback. In the magnetic world the presence of a load capacitance C on a coil introduces the magnetic impedance $\mathbb{D}=N^2C$ obeying $U=- \mathbb{D}d^2I/dt^2$. In the electrical world the presence of the capacitor creates a resonant circuit of frequency $f=\frac{1}{2\pi\sqrt{LC}}$. In the magnetic world the resonant frequency is

$$f=\frac{1}{2\pi\sqrt{\frac{\mathbb{D}}{\mathbb{R}}}}$$

7. Series/Parallel Combinations

Load impedances in parallel reflect into the magnetic domain as magnetic impedances in series, and vice versa.

8. Modelling Permanent Magnets

Permanent magnets can be modelled from their Amperian Surface Current equivalent which then appears as a mmf generator whose value is given by $U_m=B_{rem}l/\mu_0$. where l is the length of the magnet. This must be put in series with a reluctance $\mathbb{R}_m$ which is the reluctance of the *air space* occupied by the magnet. See Annex A which gives a selection of magnetic components with their electrical equivalents.

9. A Shorted Coil.

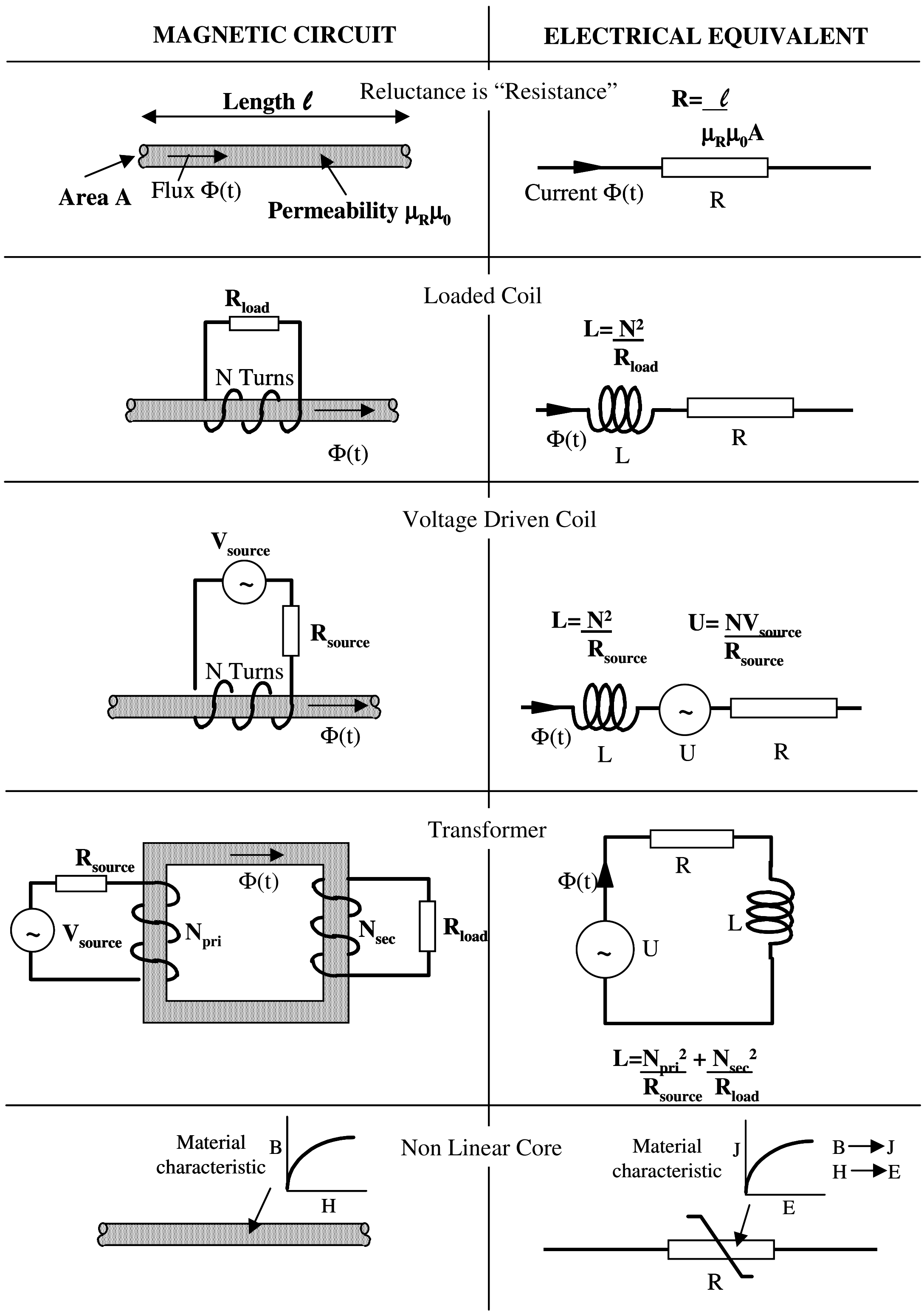
A perfectly shorted coil introduces an infinite magnetic inductance $\mathbb{L}$. In practice the coil resistance limits the inductance to a high value, but for small time spans the effect is the same, the coil acts as a *flux clamp* holding the flux at its value at the time the coil is shorted (that value could be zero). As such it is an effective magnetic switching component since if the applied flux is rising it must then divert along another path, even if that path is an external air one. Thus an actively switched short synchronized with alternating flux can be used to manipulate flux along different branches of a magnetic circuit. This type of magnetic logic has not to the Author's knowledge been used before. Only recently has it been used by an MPI researcher as a flux pump to obtain a rising staircase flux waveform where each increment comes from a low flux source (this is similar to the well known voltage multiplier in the electrical world).

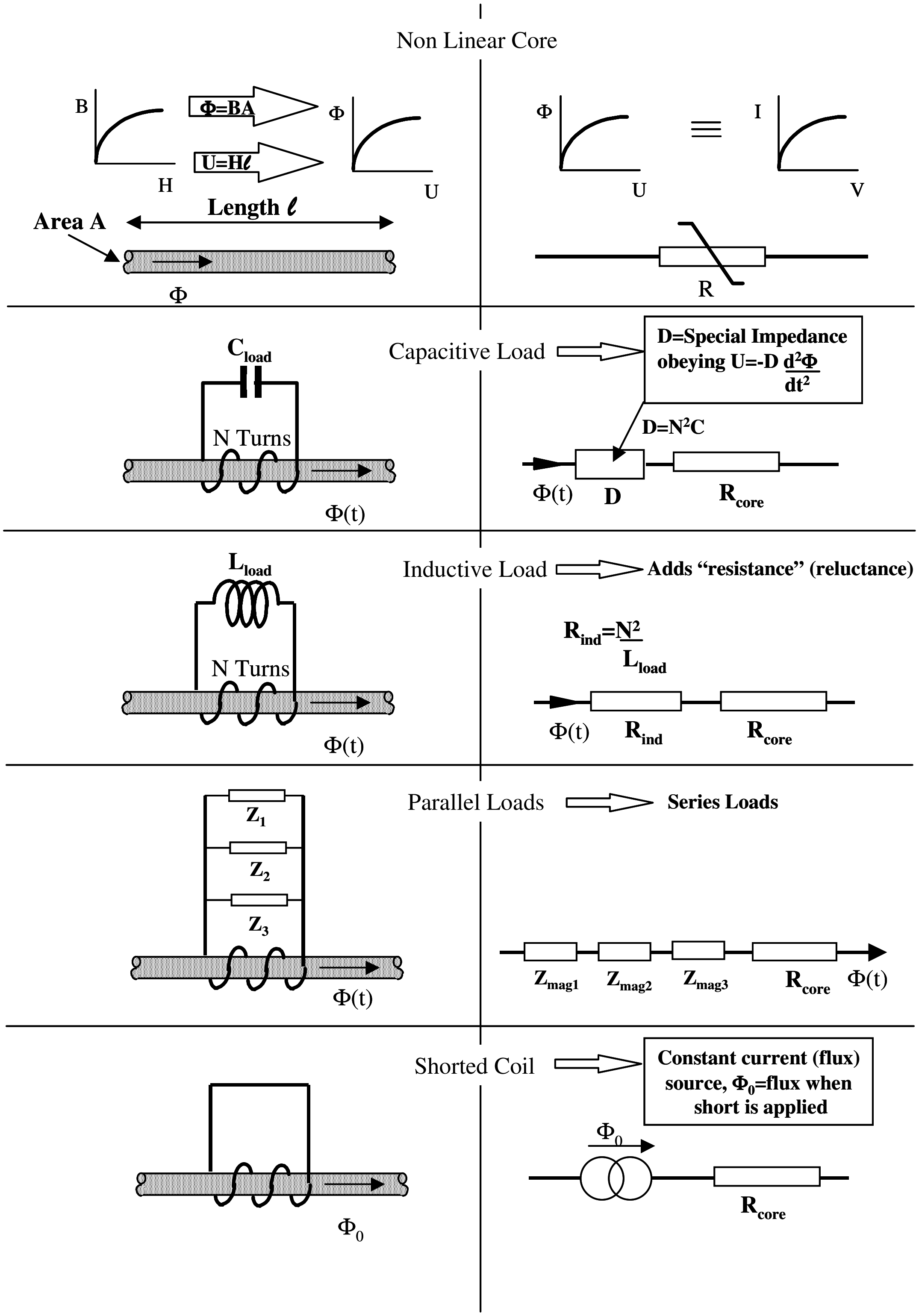
It might be noted that when a transformer is square-wave "driven" by transistor switches which connect a primary coil to a DC buss, that also acts like a shorted coil in that the DC source resistance is near zero. The flux through that coil is limited to obeying $d\Phi/dt=V_{DC}/N$ so, if there are other mmf sources trying to drive a greater flux change (such as the circulating current in a resonant tank circuit), then flux *must* be driven outside the core. The so called MEG is an example, and it might be noted that in the MEG the primary coils are alongside the magnet so that external flux is forced (by the resonant secondary circuits) to flow through part of that magnet.

10. Non-Linear Magnetic Materials.

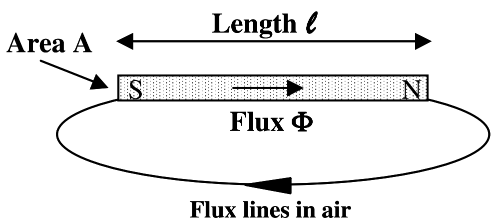
All magnetic materials are non linear in particular as they go into saturation. Modelling this effect in transformers is normally avoided, at best the shunt inductor used in the electrical model to represent the primary inductance can be given some non-linearity but this doesn't tell the full story. As we have seen the load currents do create flux in different parts of the core, and that isn't even considered in that model. However by using the magnetic domain circuits of Figures 5 or 6 the core reluctances can be made non-linear, and the circuits then analysed using SPICE like simulations.

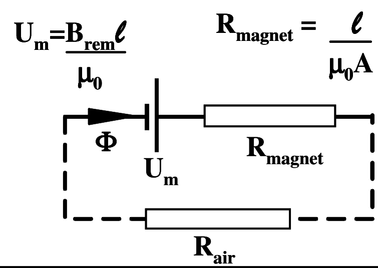
Annex A. List of Magnetic Components and their Electrical Equivalents



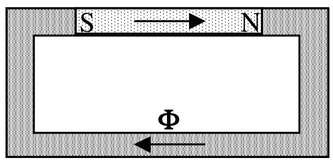

Non Linear Core
B
H
Φ=BA
U=Hℓ
Φ
U
Length ℓ
Area A
Φ
Φ
U
≡
I
V
R
Capacitive Load
D=Special Impedance obeying U=-D d²Φ/dt²
C load
N Turns
Φ(t)
D=N²C
Φ(t)
D
R core
Inductive Load
Adds "resistance" (reluctance)
L load
N Turns
Φ(t)
R ind = N²/L load
Φ(t)
R ind
R core
Parallel Loads
Series Loads
Z1
Z2
Z3
Φ(t)
Z mag1
Z mag2
Z mag3
R core
Φ(t)
Shorted Coil
Constant current (flux) source, Φ0=flux when short is applied
Φ0
Φ0
R core

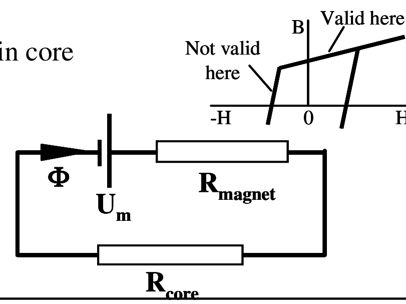
Permanent Magnet in air





Permanent Magnet in core





Example Circuit (MEG)